# GROUPS WITH ATOMIC REGULAR REPRESENTATION

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## 1. Introduction

In 1976, Larry Baggett and I studied the vanishing of matrix coefficients of representations of locally compact groups. This problem led us to consider groups with the property that their left regular representation decomposes as a direct sum of irreducible unitary representations (in this article, representation always means continuous unitary representation); that is, groups with atomic regular representation. Of course, compact groups have atomic regular representation, but we were concerned about the non-compact ones. One may feel that such groups are rare and, in a generic sense, they are. However, there are naturally occurring constructions that result in non-compact groups with atomic regular representations. Moreover, the resulting groups play a role in understanding two seemingly independent topics: construction of projections in the $L^1$-group algebra and generalizing the continuous wavelet transform to $\mathbb{R}^n$ or more general locally compact abelian groups. This survey introduces groups with atomic regular representations, or [AR] groups and describes a number of such topics that are of personal interest to me.

Since many readers of this volume may not be familiar with all of the notation and terminology of abstract harmonic analysis, we begin with an introduction to the basic spaces and the theory of representations. In Section 3, the Fourier and Fourier-Stieltjes algebras are defined. Section 4 is devoted to square-integrable representations and the powerful theorem of Duflo and Moore. We then formally define [AR] groups and provide a recipe for cooking up a variety of [AR] groups in Section 5.

The Riemann-Lebesgue Theorem tells us that the Fourier transform of an integrable function vanishes at infinity. This holds on any locally compact abelian, LCA, group and even has useful generalizations to non-abelian groups. However, on many LCA groups there exist finite Borel measures that are singular with respect to Haar measure, but whose Fourier-Stieltjes transform still vanishes at infinity. Think about the rotation invariant measure on the unit circle in the plane and what happens to a wave that scatters off a circular object and dissipates at infinity. In Section 6, this issue is formulated and the work with Baggett is introduced. The final three sections are devoted to other topics where the unique properties of [AR] groups turned out to be essential.

To keep some statements clean, I make the assumption that any locally compact group considered is second countable. In most cases, this assumption is unnecessary. No proofs are provided.

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## 2. Preliminaries

There is a veritable menagerie of spaces that arise in abstract harmonic analysis and much interest centers around the inter-relationships among the spaces. A good reference for the basic concepts introduced in this section is [11].

Let $G$ be a locally compact group equipped with left Haar measure. Integration with respect to this measure is denoted by $\int_G f(x)dx$ for any function $f$ on $G$ for which the integration makes sense. As a locally compact space, $G$ carries the following function spaces: $C(G)$, the continuous complex-valued functions on $G$; $C_b(G)$, the bounded elements of $C(G)$; $C_0(G)$, those that vanish at infinity; and $C_{00}(G)$, the elements of $C(G)$ with compact support. Both $C_b(G)$ and $C_0(G)$ are Banach spaces when equipped with the supremum norm $||\cdot||_\infty$, where $||f||_\infty = \sup\{|f(x)| : x \in G\}$ for $f : G \to \mathbb{C}$. Of course, $C_{00}(G)$ is dense in $C_0(G)$ with respect to $||\cdot||_\infty$-convergence.

For $f \in C_{00}(G)$, $\int_G f(x)dx$ is well-defined and for any $y \in G$, $\int_G f(yx)dx = \int_G f(x)dx$. This is the left invariance of the Haar integral. There exists a continuous homomorphism $\Delta_G : G \to \mathbb{C}$, called the *modular function* of $G$, such that

$$\Delta_G(y)\int_G f(xy)dx = \int_G f(x)dx, \tag{1}$$

for all $f \in C_{00}(G)$ and $y \in G$. The modular function also helps us with inversion of the variable of integration: $\int_G f(x)dx = \int_G f(x^{-1})\Delta_G(x^{-1})dx$. The group $G$ is called *unimodular* when $\Delta_G = 1$.

For any $1 \le p < \infty$, define the *$L^p$-norm* on $C_{00}(G)$ by $||f||_p = (\int_G |f(x)|^p dx)^{1/p}$, for all $f \in C_{00}(G)$. Let $L^p(G)$ denote the completion of $(C_{00}(G), ||\cdot||_p)$ as a normed linear space. As usual, the elements of $L^p(G)$ are treated as functions in their own right. For $h, k \in L^2(G)$, define $\langle h, k\rangle = \int_G h(x)\overline{k(x)}dx$. With this inner product, $L^2(G)$ is a Hilbert space. For $f \in L^1(G)$ and $g \in L^p(G), 1 \le p < \infty$, the *convolution* $f * g$ of $f$ and $g$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \tag{2}$$

which converges for almost every $x \in G$. Then $f * g \in L^p(G)$ and $||f * g||_p \le ||f||_1||g||_p$. In particular, $L^1(G)$ is a Banach algebra under convolution. There is an isometric involution, $f \to f^*$ on $L^1(G)$ given by $f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}, x \in G$.

A (continuous, unitary) *representation of* $G$ is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a Hilbert space and $\pi$ is a homomorphism of $G$ into the group of unitary operators on $\mathcal{H}_\pi$ which is continuous with respect to the weak operator topology. Often the representation may be just named as $\pi$ with $\mathcal{H}_\pi$ then assumed. If $\pi$ is a representation of $G$ and if $\xi, \eta \in \mathcal{H}_\pi$, define the *matrix coefficient function* $\varphi^\pi_{\xi,\eta}$ on $G$ by

$$\varphi^\pi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta\rangle, \tag{3}$$

for $x \in G$.

The weak operator continuity requirement on representations simply means that each matrix coefficient function $\varphi^\pi_{\xi,\eta}$ is continuous. Since $|\langle \pi(x)\xi, \eta\rangle| \le ||\eta|| \cdot ||\xi||$, $\varphi^\pi_{\xi,\eta} \in C_b(G)$, for all representations $\pi$ of $G$ and any $\xi, \eta \in \mathcal{H}_\pi$.

If $\pi$ is a representation of $G$ and $A$ is a subset of $\mathcal{H}_\pi$, then $A$ is called $\pi$-*invariant* if $\pi(x)A \subseteq A$, for all $x \in G$. For $\xi \in \mathcal{H}_\pi$, $\pi(G)\xi = \{\pi(x)\xi : x \in G\}$ is $\pi$-invariant and so is $\mathcal{K}_\xi$, the closed linear span of $\pi(G)\xi$. If there exists a $\xi \in \mathcal{H}_\pi$ such that $\mathcal{K}_\xi = \mathcal{H}_\pi$, then $\pi$ is called a *cyclic representation of* $G$ and $\xi$ a *cyclic vector for* $\pi$.

A representation $\pi$ of $G$ is called *irreducible* if $\{0\}$ and $\mathcal{H}_\pi$ are the only $\pi$-invariant closed subspaces of $\mathcal{H}_\pi$. It is easy to see that $\pi$ is irreducible if and only if every nonzero vector $\xi$ in $\mathcal{H}_\pi$ is a cyclic vector for $\pi$. When expressed in terms of matrix coefficients, this becomes a useful criterion to test for irreducibility of a given representation. We include it with two other standard characterizations in the next proposition that are essentially Schur's Lemma. Note that $\mathcal{B}(\mathcal{H}_\pi)$ denotes the space of bounded linear operators on $\mathcal{H}_\pi$ and $I$ denotes the identity operator on $\mathcal{H}_\pi$. As an algebra, the center of $\mathcal{B}(\mathcal{H}_\pi)$ is $\mathbb{C}I = \{\alpha I : \alpha \in \mathbb{C}\}$.

**Proposition 1.** *Let $\pi$ be a representation of $G$. Then, the following are equivalent:*
(a) $\pi$ *is irreducible.*
(b) *The weak operator closure of the linear span of* $\{\pi(x) : x \in G\}$ *is* $\mathcal{B}(\mathcal{H}_\pi)$.
(c) $\{T \in \mathcal{B}(\mathcal{H}_\pi) : T\pi(x) = \pi(x)T, \text{ for all } x \in G\} = \mathbb{C}I$.
(d) $\varphi^\pi_{\xi,\eta} \neq 0$, *for all* $\xi, \eta \in \mathcal{H}_\pi \setminus \{0\}$.

If $\pi$ and $\sigma$ are two representations of $G$ such that there exists a unitary map $U : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ with
$$U\pi(x) = \sigma(x)U,$$
for all $x \in G$, then $\pi$ and $\sigma$ are called *equivalent* representations. Let $[\pi]$ denote the class of all representations equivalent to $\pi$. Let
$$\widehat{G} = \{[\pi] : \pi \text{ is an irreducible representation of G}\}.$$
The set $\widehat{G}$ is called the *dual space of* $G$. If $G$ is abelian, then the equivalence of (a), (b) and (c) in Proposition 1 force any irreducible representation to be one dimensional. Thus, $\widehat{G}$ consists of one-dimensional representations, or characters, and is a locally compact group in its own right under pointwise multiplication as the group product and equipped with an appropriate topology.

If $\pi$ is any representation of $G$, it can be integrated to define a map, also denoted $\pi$, of $L^1(G)$ into $\mathcal{B}(\mathcal{H}_\pi)$, the space of bounded linear operators on $\mathcal{H}_\pi$. That is, for $f \in L^1(G)$,
$$(\xi, \eta) \to \int_G f(x)\langle \pi(x)\xi, \eta\rangle dx$$
is a bounded conjugate bilinear form on $\mathcal{H}_\pi$ and, thus, there exists $\pi(f) \in \mathcal{B}(\mathcal{H}_\pi)$ so that
$$\langle \pi(f)\xi, \eta\rangle = \int_G f(x)\langle \pi(x)\xi, \eta\rangle dx \tag{4}$$
for all $\xi, \eta \in \mathcal{H}_\pi$. It is elementary to verify that $f \to \pi(f)$ is a linear map of $L^1(G)$ into $\mathcal{B}(\mathcal{H}_\pi)$ such that $||\pi(f)|| \leq ||f||_1, \pi(f * g) = \pi(f)\pi(g)$, and $\pi(f^*) = \pi(f)^*$, for all $f, g \in L^1(G)$. Therefore, to each representation $\pi$ of $G$ there corresponds a continuous homomorphism, also denoted $\pi$, of $L^1(G)$ into $\mathcal{B}(\mathcal{H}_\pi)$ that respects the involution, a so-called $*$*-representation* of $L^1(G)$.

For each $f \in L^1(G)$, define $||f||_* = \sup_\pi\{||\pi(f)||\}$, where the supremum is over all representations $\pi$ of $G$. This defines a new norm $||\cdot||_*$ on $L^1(G)$ that is dominated by $||\cdot||_1$. The *group* $C^*$*-algebra* of $G$ is the normed *-algebraic completion of $(L^1(G), ||\cdot||_*)$ and is denoted $C^*(G)$.

The *left regular representation* $\lambda$ of $G$ is defined by translations of $L^2(G)$. That is, for $x \in G$, $\lambda(x)$ is the unitary operator defined by $\lambda(x)g(y) = g(x^{-1}y)$, for all

$y \in G, g \in L^2(G)$. One checks that $\lambda$ is a representation of $G$. When integrated up to $L^1(G)$, $\lambda$ gives the module action of $L^1(G)$ on $L^2(G)$ by left convolution. That is, $\lambda(f)g = f * g$, for all $f \in L^1(G)$ and $g \in L^2(G)$.

The *reduced $C^*$-algebra* of $G$ is $C^*_\lambda(G) = \overline{\lambda(L^1(G))}$, the closure of $\{\lambda(f) : f \in L^1(G)\}$ in $\mathcal{B}(L^2(G))$. In general, $C^*_\lambda(G)$ differs from $C^*(G)$, but they agree when $G$ is an amenable group [18].

## 3. The Fourier and Fourier-Stieltjes Algebras

In [9], Eymard laid the foundations for the study of two of the most important commutative Banach algebras associated with a, not necessarily commutative, locally compact group. The *Fourier-Stieltjes Algebra* of $G$ is

$$B(G) = \{\varphi^\pi_{\xi,\eta} : \pi \,\text{is a representation of}\, G, \xi, \eta \in \mathcal{H}_\pi\}.$$

Then $B(G)$ is an algebra over $\mathbb{C}$ when equipped with pointwise defined operations. For $\varphi \in B(G)$, $f \to (\varphi, f) = \int_G f(x)\varphi(x)dx$ is a bounded linear functional when $L^1(G)$ is equipped with the $C^*$-norm, $||\cdot||_*$, so it extends to a continuous linear functional on $C^*(G)$. This identifies $B(G)$ with $C^*(G)^*$ as a vector space and gives $B(G)$ the norm

$$||\varphi|| = \sup\{|(\varphi, f)| : f \in L^1(G), ||f||_* \leq 1\}.$$

With this norm, $B(G)$ is a Banach algebra.

The *Fourier algebra* of $G$ is $A(G) = \{\varphi^\lambda_{g,h} : g, h \in L^2(G)\}$. Eymard [9] proved that $A(G)$ is a closed ideal in $B(G)$ and identified it with the predual of the von Neumann algebra generated by the left regular representation of $G$. More precisely, let $VN(G)$ denote the weak operator topology closed subalgebra of $\mathcal{B}(L^2(G))$ generated by $\{\lambda(x) : x \in G\}$. Each $T \in VN(G)$ defines a bounded linear functional on $A(G)$ such that $(T, \varphi^\lambda_{g,h}) = \langle Tg, h\rangle$, for $g, h \in L^2(G)$. As an algebra of functions on $G$, $A(G)$ is a uniformly dense subalgebra of $C_0(G)$.

Both $A(G)$ and $B(G)$ are Banach algebras with extremely complicated structure and they have been two of the motivating examples in the promising development of the theory of operator spaces [8].

When $G$ is abelian, with dual group $\widehat{G}$, $A(G) = \{\widehat{f} : f \in L^1(\widehat{G})\}$ and $B(G) = \{\widehat{\mu} : \mu \in M(\widehat{G})\}$, where $M(\widehat{G})$ is the measure algebra of $\widehat{G}$ and the Fourier (resp. Fourier-Stieltjes) transform is an isometric isomorphism. Moreover, if $\mathcal{P} : L^2(\widehat{G}) \to L^2(G), \xi \to \widehat{\xi}$ is the Plancherel transform, then $VN(G) = \mathcal{P}L^\infty(\widehat{G})\mathcal{P}^{-1}$.

## 4. Square-integrable Representations

Let $\pi$ be an irreducible representation of $G$. An element $\xi \in \mathcal{H}_\pi$ is called an *admissible vector* if there exists a nonzero $\eta \in \mathcal{H}_\pi$ such that $\varphi^\pi_{\xi,\eta} \in L^2(G)$. If there exists a nonzero admissible vector $\xi \in \mathcal{H}_\pi$, then $\pi$ is called *square-integrable*. In that case, the set of all admissible vectors is a dense subspace $\mathcal{D}_\pi$ of $\mathcal{H}_\pi$ and, for $\xi \in \mathcal{D}_\pi$, $\varphi^\pi_{\xi,\eta} \in L^2(G)$, for all $\eta \in \mathcal{H}_\pi$. The following significant theorem was established in [7].

**Theorem 1.** (Duflo and Moore) *Let $\pi$ be a square-integrable representation of $G$. Then there exists a unique operator $K$ on $\mathcal{H}_\pi$ that is self-adjoint positive ($K$ may* *be unbounded) and such that* (i) $\mathrm{dom} K^{-1/2} = \mathcal{D}_\pi$, (ii) $\pi(x)K\pi(x)^{-1} = \Delta_G(x)^{-1}K$, *for all* $x \in G$, *and* (iii) *for* $\xi_1, \xi_2 \in \mathcal{D}_\pi, \eta_1, \eta_2 \in \mathcal{H}\pi$,

$$\langle \varphi^\pi_{\xi_1,\eta_1}, \varphi^\pi_{\xi_2,\eta_2}\rangle_{L^2(G)} = \langle K^{-1/2}\xi_1, K^{-1/2}\xi_2\rangle\langle \eta_2, \eta_1\rangle.$$

If $G$ is unimodular, then $\mathcal{D}_\pi = \mathcal{H}_\pi$, $K$ is bounded and (ii) implies that $K$ is a scalar multiple of the identity. If $G$ is compact then that scalar is the dimension of $\mathcal{H}_\pi$ and (iii) is one of the orthogonality relations for irreducible representations of a compact group.

**Definition 1.** The operator $K$ in Theorem 1 is called the *generalized dimension* of $\pi$.

If $\pi$ is a square-integrable representation of $G$, select an admissible vector $\xi$ such that $||K^{-1/2}\xi|| = 1$. Define a linear map $V_\xi : \mathcal{H}_\pi \to L^2(G)$ by

$$V_\pi\eta(x) = \overline{\varphi^\pi_{\xi,\eta}(x)} = \langle \eta, \pi(x)\xi\rangle.$$

Let $\mathcal{V}_{\pi,\xi} = \{V_\pi\eta : \eta \in \mathcal{H}_\pi\}$. Then Theorem 1(iii) implies that $\mathcal{V}_{\pi,\xi}$ is a closed subspace of $L^2(G)$ and $V_\xi$ is a unitary map of $\mathcal{H}_\pi$ onto $\mathcal{V}_{\pi,\xi}$. Also, $\mathcal{V}_{\pi,\xi}$ is $\lambda$-invariant and $V_\xi$ establishes the equivalence of $\pi$ with $\lambda$ restricted to $\mathcal{V}_{\pi,\xi}$. Thus, any square-integrable representation sits as a subrepresentation of the regular representation. On the other hand, if $\sigma$ is an irreducible subrepresentation of $\lambda$ then $\sigma$ is square-integrable.

Most noncompact groups have no square-integrable representations. However, there exist important noncompact groups, such as $SL(2,\mathbb{R})$ and the affine group of $\mathbb{R}$, which have some square-integrable representations. For $SL(2,\mathbb{R})$, the set of square-integrable representations is known as the Discrete Series and the regular representation of $SL(2,\mathbb{R})$ is the direct sum of an atomic part (which is a sum of Discrete Series Representations) and a continuous part (which has no irreducible subrepresentations). In the case of the affine group,

$$G_{\mathrm{aff}} = \{(b,a) : a, b \in \mathbb{R}, a > 0\}$$

with group product given by $(b_1, a_1)(b_2, a_2) = (b_1 + a_1b_2, a_1a_2)$, there are two square-integrable representations $\pi^+$ and $\pi^-$ and $\lambda$ is equivalent to $\aleph_0\pi^+ \oplus \aleph_0\pi^-$. This latter notation means that there are two families $\{\mathcal{K}_i^+ : i = 1, 2, 3, \cdots\}$ and $\{\mathcal{K}_i^- : i = 1, 2, 3, \cdots\}$ of mutually orthogonal closed subspaces of $L^2(G_{\mathrm{aff}})$, each $\lambda$-invariant, so that

$$L^2(G_{\mathrm{aff}}) = \sum_{i=1}^{\infty}\mathcal{K}_i^+ + \sum_{i=1}^{\infty}\mathcal{K}_i^-$$

and $\lambda$ acting on $\mathcal{K}_i^\pm$ is equivalent to $\pi^\pm$.

## 5. [AR] Groups

**Definition 2.** A locally compact group $G$ is called an [AR] group if the left regular representation of $G$ is the direct sum of irreducible representations.

Of course, any compact group is [AR] and we just saw that $G_{\mathrm{aff}}$ is an [AR] group. Actually, $G_{\mathrm{aff}}$ is representative of a large class of examples of [AR] groups that can be constructed by the following procedure.

Let $A$ be an abelian locally compact group and let $H$ be another locally compact group such that there is a homomorphism $\alpha : H \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of automorphisms of $A$. Further, assume that $(a,h) \to \alpha_h(a)$ is continuous from $A \times H$ into $A$. Form the semidirect product $A \rtimes H = \{(a,h) : a \in A, h \in H\}$, where

$$(a_1, h_1)(a_2, h_2) = (a_1\alpha_{h_1}(a_2), a_1 a_2)$$

gives the group product. The, so-called, Mackey Machine can be used to parameterize $\widehat{A \rtimes H}$ when a specific regularity assumption is satisfied. (See Section 6.6 of [11] for a readable introduction to the Mackey Machine.) We need to develop a little notation.

Since $A$ is abelian, $\widehat{A}$ is the group of characters of $A$. The action of $H$ on $A$ generates an action $(h,\chi) \to h \cdot \chi$ of $H$ on $\widehat{A}$ defined by

$$h \cdot \chi(a) = \chi(\alpha_{h^{-1}}(a)),$$

for $a \in A$. For $\chi \in \widehat{A}$, the *orbit* of $\chi$ in $\widehat{A}$ is $\mathcal{O}_\chi = \{h \cdot \chi : h \in H\}$ and the *stabilizer* of $\chi$ in $H$ is $H_\chi = \{h \in H : h \cdot \chi = \chi\}$, which is a closed subgroup of $H$. The regularity assumption that is needed has a simple formulation when we are assuming all locally compact groups considered are second countable. If there exists a Borel measurable subset $\Gamma \subseteq \widehat{A}$ so that $\mathcal{O}_\chi \cap \Gamma$ is a singleton for each orbit $\mathcal{O}_\chi$ in $\widehat{A}$, then we say that the action of $H$ on $\widehat{A}$ is *regular*. That is, the action is regular when the orbit space $\widehat{A}/H$ has a Borel cross-section. Among the consequences of this assumption are that each orbit in $\widehat{A}$ is open in its closure and the map $hH_\chi \to h \cdot \chi$ is a homeomorphism of $H/H_\chi$ with $\mathcal{O}_\chi$. To obtain [AR] groups, we consider two specific properties (let $|S|$ denote the Haar measure of a measurable $S \subseteq \widehat{A}$):

(I) There exists a countable family $\{\mathcal{O}_j : j \in J\}$ of $H$-orbits in $\widehat{A}$ such that

$$|\mathcal{O}_j| > 0, \text{ for } j \in J, \text{ and } |\widehat{A} \setminus [\cup_{j \in J} \mathcal{O}_j]\,| = 0.$$

(II) With (I) holding, for each $j \in J$ and $\chi \in \mathcal{O}_j$, $H_\chi$ is [AR].

The following theorem is an easy application of standard techniques in the Mackey Machine and is contained in the first section of [3].

**Theorem 2.** (Baggett and Taylor) *If properties* (I) *and* (II) *hold for the action of a locally compact group $H$ on an abelian group $A$, then $A \rtimes H$ is an* [AR] *group.*

Using this construction a variety of examples of noncompact [AR] groups are obtained in [3]. All of the [AR] groups highlighted there were nonunimodular and connected. In a similar manner, Mauceri and Picardello constructed families of unimodular [AR] groups in [16]; however, their examples were all totally disconnected. In [2], Baggett studied the interplay of unimodularity and [AR]. By combining his remark after Proposition 1.2 [2] with Theorem 2.3 [2], one obtains the following limitation.

**Theorem 3.** (Baggett) *If $G$ is a connected unimodular* [AR] *group, then $G$ is compact.*

Thus, if one wants to have a noncompact connected [AR] group, then one has to deal with the modular function. However, it is actually the modular function that plays a key role in many of the most interesting phenomena that occur on [AR] groups.

Of course $G_{\text{aff}}$ is an obvious example of the above construction. One already gets an interesting variety of additional examples by taking $A = \mathbb{R}^2$ and $H$ to be almost any 2-dimensional closed subgroup of the general linear group $GL(2,\mathbb{R})$. One only has to be sure that the generic orbits of $H$ do not collapse. An illustrative family of examples was studied in [19]. For $-1 \leq p \leq 1$, let

$$H_p = \left\{ \begin{pmatrix} a^{p+1} & x \\ 0 & a \end{pmatrix} : a, x \in \mathbb{R}, a > 0 \right\}.$$

Each group $\mathbb{R}^2 \rtimes H_p$ has a normal subgroup isomorphic to the 3-dimensional Heisenberg group $\mathbb{H}$ and is an extension of $\mathbb{H}$ by $\mathbb{R}$ acting on $\mathbb{H}$ as 'dilations'. More precisely, realize $\mathbb{H}$ as $\{[x,y,z] : x,y,z \in \mathbb{R}\}$ with product

$$[x_1,y_1,z_1][x_2,y_2,z_2] = [x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2].$$

A *dilation* action of $\mathbb{R}$ on $\mathbb{H}$ is one of the form $\alpha_{r,s}(t)[x,y,z] = [e^{rt}x, e^{st}y, e^{(r+s)t}z]$, where $r$ and $s$ are fixed real parameters and $t \in \mathbb{R}$. Groups of the form $\mathbb{H} \rtimes_{\alpha_{r,s}} \mathbb{R}$ were studied and classified in [19].

**Theorem 4.** (Schulz and Taylor) *Each $\mathbb{H} \rtimes_{\alpha_{r,s}} \mathbb{R}$ is isomorphic to $\mathbb{R}^2 \rtimes H_p$, for some $-1 \leq p \leq 1$. Moreover, the groups $\mathbb{R}^2 \rtimes H_p, -1 \leq p \leq 1$ are mutually non-isomorphic and $\mathbb{R}^2 \rtimes H_p$ is an* [AR] *group if and only if $p \neq -1$.*

A good exercise for the reader is to calculate the action of $H_p$ on $\widehat{\mathbb{R}^2}$, which can be identified with $\mathbb{R}^2$, to verify that (I) and (II) hold when $-1 < p \leq 1$ and see how the open orbits collapse to lines when $p = -1$.

## 6. Representations Vanishing at Infinity

It was observed earlier that $A(G) \subseteq C_0(G) \cap B(G)$. The question of when equality holds has a long history and its investigation leads to the study of [AR] groups. Indeed, it was the attempt to characterize $G$ for which $A(G) = C_0(G) \cap B(G)$ that generated my initial interest in [AR].

In 1916, Menchoff [17] showed that there exists a singular probability measure $\mu$ on $\mathbb{T}$ such that $\widehat{\mu}(n) \to 0$ as $|n| \to \infty$. So $\widehat{\mu} \in [C_0(\mathbb{Z}) \cap B(\mathbb{Z})] \setminus A(\mathbb{Z})$. Hewitt and Zuckerman [12] proved that, for an abelian locally compact group $G$, $G$ is compact if and only if $A(G) = C_0(G) \cap B(G)$. In [10], Figà-Talamanca showed that, if $G$ is unimodular, then $A(G) = C_0(G) \cap B(G)$ implies $G$ is [AR]. Since compactness and [AR] coincide for abelian groups, Figà-Talamanca's result extends that of Menchoff and Hewitt and Zuckerman.

Larry Baggett and I turned our attention to extending Menchoff's theorem to general $G$ in [4]. We showed that, for any non-[AR] group $G$, there exists a representation $\pi$ of $G$ that has no subrepresentation in common with the regular representation and a $\xi \in \mathcal{H}_\pi, ||\xi|| = 1$ such that $\varphi^\pi_{\xi,\xi} \in C_0(G)$. This is neatly stated as the following theorem.

**Theorem 5.** (Baggett and Taylor) *Let $G$ be a second countable locally compact group. If $A(G) = C_0(G) \cap B(G)$, then $G$ is an* [AR] *group.*

One may speculate that [AR] groups are characterized by $A(G) = C_0(G) \cap B(G)$. However, in [3], we constructed an [AR] group $G$ and an irreducible representation $\pi$ of $G$ that is not square-integrable but still vanishes at infinity; that is, for $\xi \in \mathcal{H}_\pi, \varphi^\pi_{\xi,\xi} \in C_0(G)$. So, if $\xi \in \mathcal{H}_\pi \setminus \{0\}$, then $\varphi^\pi_{\xi,\xi} \in [C_0(G) \cap B(G)] \setminus A(G)$. The problem of characterizing groups for which $A(G) = C_0(G) \cap B(G)$ remains open.

## 7. Geometric Properties of $A(G)$ and the [AR] Property

If $G$ is abelian, then $A(G)$ is isometrically isomorphic to $L^1(\widehat{G})$. Moreover, an abelian group $G$ is [AR] if and only if $G$ is compact, equivalently, $\widehat{G}$ is discrete, equivalently, Haar measure on $\widehat{G}$ is atomic. If $(X, \mu)$ is a measure space, there are a number of geometric properties of Banach spaces that hold on $L^1(X, \mu)$ if and only in $\mu$ is atomic. Essentially, these properties distinguish $l^1$ from $L^1[0, 1]$. When $G$ is non-abelian, we cannot say that $A(G)$ is $L^1(X, \mu)$ for some measure space $(X, \mu)$. However, many analogies hold. In particular, when $G$ is [AR], $A(G)$ shares characteristics with $l^1$.

One striking property that $L^1(\mu)$ has only when $\mu$ is atomic is that of being the dual of another Banach space.

**Theorem 6.** (Taylor [21]) *$G$ is an* [AR] *group if and only if $A(G)$ is a dual Banach space.*

**Definition 3.** A Banach space $E$ has the *Radon-Nikodym Property*, RNP, if whenever $(X, \mu)$ is a finite measure space and $\nu$ is a $\mu$-continuous vector-valued measure from $X$ into $E$ of bounded variation, there exists a Bochner integrable $g$ from $X$ into $E$ such that $\nu(a) = \int_A g d\mu$ for every measurable $A \subseteq X$.

The book by Diestel and Uhl [6] is a good source of information on RNP. There one will find a long list of properties of Banach spaces that turn out to be equivalent to RNP. For a measure space $(X, \mu)$, $L^1(\mu)$ has the RNP if and only if $\mu$ is an atomic measure.

**Theorem 7.** (Taylor [21]) *$G$ is an* [AR] *group if and only if $A(G)$ has the RNP.*

## 8. Constructing Projections in $L^1(G)$

A self-adjoint idempotent in the Banach *-algebra $L^1(G)$ is called a *projection*. That is, $f \in L^1(G)$ is a projection if $f * f = f = f^*$. This is the case if and only if $\pi(f)$ is a projection operator on $\mathcal{H}_\pi$, for every irreducible representation $\pi$ of $G$. Since the left regular representation is faithful as a representation of $L^1(G)$, $f$ is a projection if and only if $\lambda(f)$ is a projection operator on $L^2(G)$. In [13] and [14], we developed methods for deciding whether certain groups $G$ admit nonzero projections in $L^1(G)$ and for explicit constructions of projections when they can exist. The most satisfying results apply for the type of [AR] groups constructed in Section 5.

To keep things simple, we will assume that $G = A \rtimes_\alpha H$ with $A$ abelian and $H$ acting in such a manner that there are $\chi_1, \cdots, \chi_n \in \widehat{A}$ such that, for $1 \leq i \leq n$, $H_{\chi_i} = \{e\}$, $\mathcal{O}_{\chi_i}$ is open, and $|\widehat{A} \setminus [\cup_i \mathcal{O}_{\chi_i}]| = 0$. Then clearly (I) and (II) of Section 5 hold and $G$ is an [AR] group. Constructing projections in $L^1(G)$ requires some preparation.

Let $\Delta_H$ denote the modular function of $H$ and let $\delta(h)$ denote the modulus of the automophism $\alpha(h)$ of $A$. Thus,

$$\delta(h) \int_A g(\alpha_h(a)) da = \int_A g(a) da,$$

for any $g \in C_{00}(A)$, for example. There is a natural unitary representation $\rho$ of $G$ on $L^2(A)$ given by

$$\rho(a, h) g(b) = \delta(h)^{-1/2} g(\alpha_{h^{-1}}(a^{-1}b)), \tag{5}$$

for all $b \in A, g \in L^2(A)$, and $(a,h) \in G$. To decompose $\rho$ into irreducibles, let $\mathcal{F} : L^2(A) \to L^2(\widehat{A})$ denote the Fourier transform as a unitary map. Define $\pi(a,h) = \mathcal{F}\rho(a,h)\mathcal{F}^{-1}$, for $(a,h) \in G$. Then $\pi$ is a representation equivalent to $\rho$ and a short calculation using standard properties of the Fourier transform shows that

$$\pi(a,h)\xi(\chi) = \delta(h)^{1/2}\chi(a)\xi(h^{-1} \cdot \chi), \tag{6}$$

for $\chi \in \widehat{A}, \xi \in L^2(\widehat{A})$, and $(a,h) \in G$. Since each $\mathcal{O}_{\chi_i}$ is open, we can restrict Haar measure on $\widehat{A}$ to $\mathcal{O}_{\chi_i}$ and consider $L^2(\mathcal{O}_{\chi_i})$ as a closed subspace of $L^2(\widehat{A})$ in the obvious way. Since $\mathcal{O}_{\chi_i}$ is a $H$-invariant set in $\widehat{A}$, one sees from (6) that $L^2(\mathcal{O}_{\chi_i})$ is a $\pi$-invariant subspace of $L^2(\widehat{A})$. Define $\pi_i(a,h) = \pi(a,h)_{|L^2(\mathcal{O}_{\chi_i})}$, for $(a,h) \in G$ and for $1 \leq i \leq n$. It can be shown that each $\pi_i$ is irreducible and, in fact, equivalent to $\mathrm{ind}_A^G\chi_i$, the representation of $G$ induced from $\chi_i$.

If $\mathcal{H}_i^2 = \{f \in L^2(A) : \mathcal{F}(f) \in L^2(\mathcal{O}_{\chi_i})\}$, then $L^2(A) = \sum_i^{\oplus} \mathcal{H}_i^2$ and each $\mathcal{H}_i^2$ is $\rho$-invariant. Let $\rho_i$ be the subrepresentation of $\rho$ associated with $\mathcal{H}_i^2$. Note that $\rho_i$ is equivalent to $\pi_i$, for $1 \leq i \leq n$. Not only is $\rho = \sum_i^{\oplus} \rho_i$, but the left regular representation $\lambda$ is equivalent to $\sum_i^{\oplus} \aleph_0 \cdot \rho_i$.

In analogy with the projections that arise from the orthogonality relations associated with irreducible representations of compact groups, one can hope that there might be projections associated with the square-integrable representations, $\rho_i$, in light of Theorem 1. Investigating this potential in [13] and [14] led to defining, for $w \in L^2(A)$,

$$f_w(a,h) = \left[\frac{\delta(h)}{\Delta_H(h)}\right]^{1/2} \langle w, \rho_i(a,h)w\rangle, \tag{7}$$

**Definition 4.** A *projection generating function* (PGF) associated with $\mathcal{O}_{\chi_i}$ is a $w \in L^2(A)$ which satisfies:

(i) $\mathrm{supp}(\widehat{w}) \subseteq \mathcal{O}_{\chi_i}$
(ii) $\int_H |\widehat{w}(h^{-1} \cdot \chi_i)|^2 dh = 1$
(iii) $f_w \in L^1(G)$, with $f_w$ defined by (7).

For a projection $f$ in $L^1(G)$ the support of $f$ in $\widehat{G}$ is $s(f) = \{\sigma \in \widehat{G} : \sigma(f) \neq 0\}$. For projections $f$ and $g$ in $L^1(G)$ we write $f \leq g$ if $f * g = f$. A projection $f$ in $L^1(G)$ is called *minimal* if $f \neq 0$ and, for any projection $g$ in $L^1(G)$, $g \leq f$ implies $g = 0$ or $g = f$.

**Theorem 8.** (Kaniuth and Taylor [14]) *With the above notation, the following three facts hold:*

(a) *Let $w$ be a* PGF *associated with the orbit $\mathcal{O}_{\chi_i}$. Then $f_w$, as defined in (7), is a minimal projection.*

(b) *Every minimal projection in $L^1(G)$ is of the form $f_w$, with $w$ a* PGF *associated with $\mathcal{O}_{\chi_i}$, for some $1 \leq i \leq n$.*

(c) *Every projection in $L^1(G)$ with $s(f) = \rho_i$ is the orthogonal direct sum of minimal projections.*

Many questions remain open on how completely one may describe the projections in $L^1(G)$ for a general [AR] group $G$.

## 9. Constructing Continuous Wavelet Transforms on $\mathbb{R}^n$

The final area to mention in which [AR] groups play a central role is the general construction of continuous wavelet transforms of $L^2(\mathbb{R}^k)$ or $L^2(A)$ for a general locally compact abelian group $A$. Again, as in the previous section, assume that $G = A \rtimes_\alpha H$ with $A$ abelian and $H$ acting in such a manner that there are $\chi_1, \cdots, \chi_n \in \widehat{A}$ such that, for $1 \leq i \leq n$, $H_{\chi_i} = \{e\}$, $\mathcal{O}_{\chi_i}$ is open, and $|\widehat{A} \setminus [\cup_i \mathcal{O}_{\chi_i}]| = 0$. The results in this section are based on [5]. There we assumed $A = \mathbb{R}^k$, but the generalization is direct.

Let $\rho$ be the representation of $G$ on $L^2(A)$ defined in (5). For $w \in L^2(A)$ and $(a, h) \in G$, define $w_{a,h} = \rho(a, h)w$.

For each $i \in \{1, 2, \cdots, n\}$, let $\rho_i$ be the irreducible subrepresentation of $\rho$ associated with $\mathcal{H}_i^2$. Since $\rho_i$ is a square-integrable representation, the theorem of Duflo and Moore applies. A key point is to identify the operator $K$ that arises in Theorem 1. This operator comes from the relationship between two natural measures on the orbit $\mathcal{O}_{\chi_i}$. Since $h \to h \cdot \chi_i$ is a homeomorphism of $H$ with $\mathcal{O}_{\chi_i}$, we can transfer the left Haar measure of $H$ to $\mathcal{O}_{\chi_i}$. Call this measure $\nu_i$. Let $\mu_i$ denote Haar measure on $\widehat{A}$ restricted to $\mathcal{O}_{\chi_i}$. Let $\Psi_i = [d\mu_i/d\nu_i]$, the Radon-Nikodym derivative. Then $\Psi_i$ is a positive continuous function on $\mathcal{O}_{\chi_i}$. Therefore, $\Psi_i$ operates on $L^2(\mathcal{O}_{\chi_i})$ by pointwise multiplication as an unbounded operator with domain $\{\xi \in L^2(\mathcal{O}_{\chi_i}) : \Psi\xi \in L^2(\mathcal{O}_{\chi_i})\}$. Now define $K_i$ on $\mathcal{H}_i^2$ by $K_i g = \mathcal{F}^{-1}(\Psi_i \widehat{g})$, for any $g \in \mathcal{H}_i^2$ such that $\Psi_i \widehat{g} \in L^2(\mathcal{O}_{\chi_i})$ (these are the admissible vectors in the language of Section 4). Select $w^i \in \mathcal{H}_i^2$ such that $||K_i w^i||_2^2 = ||\Psi_i w^i||_2^2 = 1$.

The set $W = \{w^1, \cdots, w^n\}$ forms a multi-wavelet for a continuous wavelet transform of $L^2(A)$ in the sense expressed by (b) in the following theorem which can be derived from [5] and Theorem 1.

**Theorem 9.** (Bernier and Taylor) *With the above notation,*

(a) *$K_i$ is the generalized dimension of $\rho_i$, for $i = 1, \cdots, n$.*

(b) *For any $f \in L^2(A)$,*

$$f = \sum_{i=1}^{n} \int_A \int_H \langle f, w^i_{a,h} \rangle w^i_{a,h} \frac{dh\, da}{\delta(h)},$$

*weakly in $L^2(A)$.*

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