Wavelets with crystal symmetry shifts

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Abstract

A generalization of multi-dimensional wavelet theory is introduced in which the usual lattice of translational shifts is replaced by a discrete subgroup of the group of affine, area preserving, transformations of Euclidean space. The dilation matrix must now be compatible with the group of shifts. An existence theorem for a multiwavelet in the presence of a multiresolution analysis is established and examples are given to illustrate the theory with two dimensional crystal symmetry groups as shifts.

1 Introduction

In the classical theory of wavelets on $\mathbb{R}^n$, wavelets are transformed by shifts, selected from a discrete cocompact subgroup of the translation group, and the powers of an appropriate dilation matrix. The recently developing theory of composite dilations, see [4], [5], [9], and [3], demonstrates the value of introducing transformation by selected matrices other than powers of the dilation. In applications, so far, these additional transformations are area preserving. We feel there is value in considering all the area preserving transformations, that are used, as "shifts". The purpose of this article is to formulate the appropriate definitions of multiwavelets and generalized multiresolution analysis in this context and to demonstrate through examples how crystal symmetry groups can than be incorporated as the group of shifts.

We find it necessary to formulate a theory based on this more general concept of shifts because some of the important crystal groups do not split as a semidirect product of the translation subgroup and a finite point group. Such non-splitting groups are called nonsymmorphic. In two dimensions, four of the seventeen wallpaper groups are nonsymmorphic while, in three dimensions, 146 out of the 219 crystal groups are nonsymmorphic. These nonsymmorphic groups are not covered by the theory of composite dilations.

In section 2, we establish the notation we will use and define multiwavelets in the new setting. Generalized multiresolution analysis and an appropriate concept of scaling ensemble is introduced in section 3 where the main theorem on the existence of a multiwavelet is established. After introducing the definition of crystal groups in 4, we devote section 5 to constructing the appropriate ingredients in several examples. The article ends with some concluding remarks.

2 Preliminaries

Let $\mathrm{GL}_n(\mathbb{R})$ denote the group of invertible $n \times n$ real matrices. If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is any affine map then there are unique $x \in \mathbb{R}^n$ and $L \in \mathrm{GL}_n(\mathbb{R})$ such that

$\varphi(z) = L(z+x)$, for all $z \in \mathbb{R}^n$. (Elements of $\mathbb{R}^n$ will be considered as column vectors). We use the notation $[x, L]$ for φ and $[x, L] \cdot z = L(z+x)$. The affine group of $\mathbb{R}^n$ is the set of all affine maps with composition as the group product. We denote it by

$$\mathrm{Aff}_n(\mathbb{R}) = \{[x, L] : x \in \mathbb{R}^n, L \in \mathrm{GL}_n(\mathbb{R})\}.$$

$\mathrm{Aff}_n(\mathbb{R})$ is then a semidirect product of the additive group $\mathbb{R}^n$ by $\mathrm{GL}_n(\mathbb{R})$ with the natural action. For $[x, L], [y, M] \in \mathrm{Aff}_n(\mathbb{R})$, calculating the composition gives

$$[x, L][y, M] = [M^{-1}x + y, LM].$$

If I denotes the identity $n \times n$ matrix, then $[0, I]$ is the identity in $\mathrm{Aff}_n(\mathbb{R})$ and $[x, L]^{-1} = [-Lx, L^{-1}]$.

If $E \subseteq \mathbb{R}^n$ is a Lebesgue measurable set with Lebesgue measure $|E|$ then, for $[x, L] \in \mathrm{Aff}_n(\mathbb{R})$,

$$|[x, L] \cdot E| = |\det L||E|.$$

Thus, the measure preserving affine maps are all of the form $[x, L]$ with $L \in \widetilde{\mathrm{SL}}_n(\mathbb{R})$ where

$$\widetilde{\mathrm{SL}}_n(\mathbb{R}) = \{L \in \mathrm{GL}_n(\mathbb{R}) : |\det L| = 1\}.$$

Let $\mathrm{SAff}_n(\mathbb{R}) = \{[x, L] : x \in \mathbb{R}^n, L \in \widetilde{\mathrm{SL}}_n(\mathbb{R})\} = \mathbb{R}^n \rtimes \widetilde{\mathrm{SL}}_n(\mathbb{R})$.

For $[x, L] \in \mathrm{Aff}_n(\mathbb{R})$, $f \in L^2(\mathbb{R}^n)$ and $z \in \mathbb{R}^n$, let

$$\begin{aligned} \pi[x, L]f(z) &= |\det L|^{-1/2} f([x, L]^{-1} \cdot z) \\ &= |\det L|^{-1/2} f(L^{-1}z - x). \end{aligned}$$

Then π is a unitary representation of $\mathrm{Aff}_n(\mathbb{R})$ and provides convenient notational shortcuts in the formulation of wavelet theory and its variations.

We take the point of view in this article that general wavelet theory refers to any form of analysis or synthesis involving subsets $\mathcal{S} \subset \mathrm{Aff}_n(\mathbb{R})$ and $\mathcal{W} \subset L^2(\mathbb{R}^n)$ so that, for any given $f \in L^2(\mathbb{R}^n)$, the set of coefficients

$$\{\langle f, \pi[x, L]w\rangle : [x, L] \in \mathcal{S}, w \in \mathcal{W}\}$$

is useful in understanding f.

The set $\mathcal{S}$ may be of the form $\mathbb{R}^n \rtimes H$ for some closed subgroup H of $\mathrm{GL}_n(\mathbb{R})$ as investigated in [2] and [12] and successfully used in the rapidly developing theory and applications of the continuous shearlet transform (see [8] and [15]). However, in the present study we concentrate on systems built with discrete sets $\mathcal{S}$ and finite $\mathcal{W}$.

A motivating development for us was the emerging theory of wavelets with composite dilations, including discrete shearlet transforms, introduced in [4] and [5]. For a recent introduction to the theory see [9]; in particular, Section 1.4.1 of [9] is directly relevant to what follows.

For our purposes, the key ingredients in a system of wavelets with composite dilations are the following

$$\Psi_{\mathcal{AB}} = \{D_A D_B T_x \Psi \ : \ x \in \mathbb{Z}^n, B \in \mathcal{B}, A \in \mathcal{A}\}, \tag{1}$$

where $\Psi = \{\psi_1, \ldots, \psi_l\} \subset L^2(\mathbb{R}^n)$,

$$\begin{aligned} T_x f(z) &= f(z - x), \\ D_A f(z) &= |\det A|^{-1/2} f(A^{-1}z), \end{aligned}$$

and $\mathcal{A}, \mathcal{B} \subset GL_n(\mathbb{Z})$. Note that the lattice $\mathbb{Z}^n$ in (1) may be replaced by some other lattice in $\mathbb{R}^n$ and $\mathcal{A}$ and $\mathcal{B}$ be required to respect that lattice, but this is not a substantial generalization. If $\Psi_{\mathcal{AB}}$ is an ON basis for $L^2(\mathbb{R}^n)$, then Ψ is called an ON multiwavelet. We point out that

$$D_A D_B T_x f = \pi[0, A]\pi[x, B]f = \pi[x, AB]f, \tag{2}$$

for $f \in L^2(\mathbb{R}^n), x \in \mathbb{Z}^n, B \in \mathcal{B}, A \in \mathcal{A}$.

This theory is most pursued in the cases where $\mathcal{B}$ is a subgroup of $\widetilde{\mathrm{SL}}_n(\mathbb{Z})$ and $\mathcal{A} = \{A^k : k \in \mathbb{Z}\}$, where $A \in GL_n(\mathbb{Z})$ is an appropriately compatible matrix. When $\mathcal{B}$ is a subgroup of $\widetilde{\mathrm{SL}}_n(\mathbb{Z})$, let

$$\Gamma = \{[x, B] : x \in \mathbb{Z}^n, B \in \mathcal{B}\} = \mathbb{Z}^n \rtimes \mathcal{B},$$

the semidirect product of $\mathbb{Z}^n$ with the acting group $\mathcal{B}$. Then Γ is a discrete subgroup of $\mathrm{SAff}_n(\mathbb{R})$, and the compatibility requirement on A can be expressed as $[0, A]\Gamma[0, A^{-1}] \subsetneq \Gamma$ and $\Gamma/[0, A]\Gamma[0, A^{-1}]$ is finite. A useful point of view is to think of Γ as the *shift* group and A as the dilation matrix.

Although the theory of wavelets with composite dilations, as formulated in (1), is proving to be extremely useful, there are cases of interest which are closely related but not included in the current formulation. We are most concerned with including the cases where Γ is the symmetry group of some crystal structure in $\mathbb{R}^n$ (crystal group). Some crystal groups split as semidirect products and fit into the above setting, but not all. Therefore, we introduce the following definition.

Definition 1 *Let* Γ *be a discrete subgroup of* $\mathrm{SAff}_n(\mathbb{R})$ *and let* $A \in \mathrm{GL}_n(\mathbb{R})$ *be such that* $[0, A]\Gamma[0, A^{-1}] \subsetneq \Gamma$ *and* $\Gamma/[0, A]\Gamma[0, A^{-1}]$ *is finite. An* $A\Gamma$*-multiwavelet is a finite set of functions* $\Psi = \{\psi_1, \ldots, \psi_l\} \subset L^2(\mathbb{R}^n)$ *such that*

$$\{\pi[0, A^k]\pi[x, C]\psi_i : k \in \mathbb{Z}, [x, C] \in \Gamma, 1 \leq i \leq l\} \tag{3}$$

is a Parseval frame in $L^2(\mathbb{R}^n)$*.*

Recall that $\mathcal{F} \subset \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, is called a Parseval frame in $\mathcal{H}$ if, for all $\xi \in \mathcal{H}, ||\xi||^2 = \sum_{\eta \in \mathcal{F}} |\langle \xi, \eta \rangle|^2$. If the set in (3) is an orthonormal basis of $L^2(\mathbb{R}^n)$, then Ψ is called an ON-$A\Gamma$-multiwavelet. We will call A *compatible with* Γ when $[0, A]\Gamma[0, A^{-1}] \subsetneq \Gamma$ and $\Gamma/[0, A]\Gamma[0, A^{-1}]$ is finite.

The important features of Definition 1 are that it includes all the examples of composite dilation systems which have been used so far and it allows for shift groups that are more general than $\mathbb{Z}^n \rtimes \mathcal{B}$. For example, Γ may be any crystal group.

3 Existence of $A\Gamma$-multiwavelets

In [1], a quite general concept of multiresolution analysis was introduced. We specialize their definition with the goal of constructing $A\Gamma$-multiwavelets and, ultimately, efficient algorithms.

Definition 2 *Let* Γ *be a discrete subgroup of* $\mathrm{SAff}_n(\mathbb{R})$ *and let* $A \in \mathrm{GL_n}(\mathbb{R})$ *be compatible with* Γ*. A sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R}^n)$ *is called a generalized multiresolution analysis relative to* Γ *and* A $(\mathrm{GMRA}_{A\Gamma})$ *if*

(i) $\pi(\Gamma)V_0 \subseteq V_0$*,*

(ii) $V_j = \pi[0, A^{-j}]V_0$*, for all* $j \in \mathbb{Z}$*,*

(iii) $V_j \subset V_{j+1}$*, for all* $j \in \mathbb{Z}$*, and*

(iv) $\cap V_j = \{0\}$ *and* $\overline{\cup V_j} = L^2(\mathbb{R}^n)$*.*

One may expect that we introduce the stricter notion of a multiresolution analysis relative to Γ and A when there exists a scaling function in V_0; however, examples of crystal groups acting on $\mathbb{R}^2$ suggest a slightly more general concept.

Definition 3 *Let* $\{V_j\}_{j\in\mathbb{Z}}$ *be a* $\mathrm{GMRA}_{A\Gamma}$*. A subset* $\{\varphi_1, \dots, \varphi_r\} \subset V_0$ *is called a finite scaling ensemble* (FSE) *if*

$$\{\pi[x, C]\varphi_i : [x, C] \in \Gamma, 1 \leq i \leq r\}$$

is an orthonormal basis of V_0*. If a* FSE *exists, then* $\{V_j\}_{j\in\mathbb{Z}}$ *is called a multiresolution analysis relative to* Γ *and* A ($\mathrm{MRA}_{A\Gamma}$)*.*

If $\{V_j\}_{j\in\mathbb{Z}}$ is a $\mathrm{GMRA}_{A\Gamma}$, define $W_j = V_{j+1} \ominus V_j$ for $j \in \mathbb{Z}$. So $W_0 = V_1 \ominus V_0$ and $W_j = \pi[0, A^{-j}]W_0$ for $j \in \mathbb{Z}$. Observe that W_1 is invariant under $\pi[x, C]$ for all $[x, C] \in \Gamma$.

The left regular representation λ_Γ is a unitary representation of Γ which acts on $\ell^2(\Gamma)$ by left translation. That is, for $\xi \in \ell^2(\Gamma)$, $[x, C] \in \Gamma$,

$$\lambda_\Gamma[x, C]\xi([y, D]) = \xi([x, C]^{-1}[y, D]),$$

for all $[y, D] \in \Gamma$. Let δ denote the function in $\ell^2(\Gamma)$ which is 1 at the identity element, $[0, I]$, and 0 everywhere else. Then

$$\{\lambda_\Gamma[x, C]\delta : [x, C] \in \Gamma\}$$

is clearly an orthonormal basis of $\ell^2(\Gamma)$.

If $\{\varphi_1, \dots, \varphi_r\}$ is a FSE for an $\mathrm{MRA}_{A\Gamma}$ $\{V_j\}_{j\in\mathbb{Z}}$, then define

$$V_{0,i} = \mathrm{cl} < \{\pi[x, C]\varphi_i : [x, C] \in \Gamma\} >,$$

$$V_{1,i} = \pi[0, A^{-1}]V_{0,i} \text{ and } W_{0,i} = V_{1,i} \ominus V_{0,i},$$

for $1 \leq i \leq r$.

For each $i \in \{1, 2, \ldots, r\}$, define a Hilbert space isomorphism $\Phi_i : \ell^2(\Gamma) \to V_{0,i}$ by

$$\Phi_i(\lambda_\Gamma[x, C]\delta) = \pi[x, C]\varphi_i,$$

for all $[x, C] \in \Gamma$, extending by linearity and continuity. Then

$$\Phi_i \lambda_\Gamma[x, C]\xi = \pi[x, C]\Phi_i\xi,$$

for all $\xi \in \ell^2(\Gamma)$. Thus, $V_{0,i}$ is a π-invariant subspace of $L^2(\mathbb{R}^n)$ and the restriction of π to this subspace is unitarily equivalent to the left regular representation.

Now, $\pi[0, A^{-1}] : V_{0,i} \to V_{1,i}$ is a surjective isometry, so

$$\{\pi[0, A^{-1}]\pi(\gamma)\varphi_i : \gamma \in \Gamma\} = \{\pi(\omega)\pi[0, A^{-1}]\varphi_i : \omega \in [0, A^{-1}]\Gamma[0, A]\} \tag{4}$$

is an ON basis of $V_{1,i}$. Let d denote the index of Γ in $[0, A^{-1}]\Gamma[0, A]$, which is the same as the index of $[0, A]\Gamma[0, A^{-1}]$ in Γ. Let $\{[y_1, D_1], \ldots, [y_d, D_d]\}$ be a set of representatives of the cosets of Γ in $[0, A^{-1}]\Gamma[0, A]$. So

$$[0, A^{-1}]\Gamma[0, A] = \bigcup_{j=1}^{d} \Gamma[y_j, D_j],$$

a disjoint union. For $1 \leq j \leq d$, define $\varphi_{i,j} = \pi[y_j, D_j]\pi[0, A^{-1}]\varphi_i$. By (4)

$$\{\pi[x, C]\varphi_{i,j} : [x, C] \in \Gamma, 1 \leq j \leq d\}$$

is an ON basis of $V_{1,i}$. Define

$$Z_{i,j} = \text{cl} < \{\pi[x, C]\varphi_{i,j} : [x, C] \in \Gamma\} >,$$

for $1 \leq j \leq d$. Then $V_{1,i} = Z_{i,1} \oplus \cdots \oplus Z_{i,d}$, each $Z_{i,j}$ is invariant under $\pi(\Gamma)$ and the restriction of π as a representation of Γ to $Z_{i,j}$ is equivalent to λ_Γ with the same argument which was used for $V_{0,i}$. Therefore, the action of $\pi(\Gamma)$ on $V_{1,i}$ is equivalent to the direct sum of d copies of λ_Γ while the action of $\pi(\Gamma)$ on $V_{0,i}$ is equivalent to one copy of λ_Γ. With this set-up, the proof of Lemma 1.3 of [1], using the cancelation property for finite von Neumann algebras applies to prove that the action of $\pi(\Gamma)$ on $W_{0,i}$ is equivalent to $d-1$ copies of λ_Γ. Since $W_0 = W_{0,1} \oplus \cdots \oplus W_{0,r}$, we arrive at our main theorem.

Theorem 4 *Let Γ be a discrete subgroup of $\mathrm{SAff}_n(\mathbb{R})$ and let $A \in \mathrm{GL}_n(\mathbb{R})$ be compatible with Γ. Let d be the index of $[0, A]\Gamma[0, A^{-1}]$ in Γ. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an $\mathrm{MRA}_{A\Gamma}$ with a FSE, $\{\varphi_1, \ldots, \varphi_r\}$. Then there exists an ON-$A\Gamma$-multiwavelet $\Psi = \{\psi_1, \ldots, \psi_{r(d-1)}\}$.*

Proof: With the notation above, write $W_0 = X_1 \oplus \cdots \oplus X_{r(d-1)}$, where each X_i is a $\pi(\Gamma)$-invariant closed subspace on which the action of $\pi(\Gamma)$ is equivalent to λ_Γ. Let $U_i : \ell^2(\Gamma) \to X_i$ be a Hilbert space isomorphism intertwining λ_Γ with π restricted to X_i. Define $\psi_i = U_i\delta$, for $1 \leq i \leq r(d-1)$. By construction, $\{\pi[x, C]\psi_i : [x, C] \in \Gamma, 1 \leq i \leq r(d-1)\}$ is an ON basis of W_0. Thus,

$$\{\pi[0, A^k]\pi[x, C]\psi_i : k \in \mathbb{Z}, [x, C] \in \Gamma, 1 \leq i \leq r(d-1)\}$$

is an ON basis of $L^2(\mathbb{R}^n)$.

4 Crystal symmetry groups

As a major application of using more general discrete subgroups of $\mathrm{SAff}_n(\mathbb{R})$ than $\mathbb{Z}^n$ as the group of shifts in a theory of multiresolution analysis and wavelets, we focus our attention on crystal groups.

Definition 5 *A subgroup* $\Gamma \subseteq \mathrm{SAff}_n(\mathbb{R})$ *is a crystal group if it is discrete and* $\mathbb{R}^n/\Gamma$ *is compact.*

In two dimensions there are 17 crystal groups. These are sometimes called the *wallpaper groups.*

Remark 6 *If* Γ *is a crystal group, then there exists a normal subgroup* N *of* Γ *such that:*

1. N *is isomorphic to* $\mathbb{Z}^n$*;*

2. $D = \Gamma/N$ *is a finite group called the* point group*.*

It is important to note that Γ, since it has an abelian subgroup of finite index, has a very well understood representation theory. This is in sharp contrast to any discrete group which does not have an abelian subgroup of finite index as a result of Thoma's famous classification of Type I discrete groups [14]. In [13], a computationally practical Fourier transform was defined for crystal groups. This should prove to be an essential tool as we develop the theory of wavelets with crystal symmetries.

We end this paper with some examples illustrating the construction of an $\mathrm{MRA}_{A\Gamma}$ in the crystal symmetry setting. Compatible dilation matrices, $\mathrm{MRA}_{A\Gamma}$'s and associated wavelets are constructed for all 17 two dimensional crystal groups in [10] and [11]. A more in depth analysis can be found in [11].

5 Examples

We selected three particular examples from among the two dimensional crystal groups. The first example is not a semidirect product, so it is not covered by the theory of composite dilations. The second illustrates the necessity of our introduction of finite scaling ensembles, as opposed to a single scaling vector. The third example is included as the planar crystal group with the greatest symmetry; that is, the largest point group.

Given a crystal group Γ acting on $\mathbb{R}^2$, take a compatible $A \in \mathrm{GL}_n(\mathbb{Z})$ and choose a fundamental domain R. Let $\varphi = \chi_R/||R||$ and consider the sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by,

$$\begin{aligned} V_0 &= \overline{\mathrm{span}}\{\pi[x,C]\varphi \ : \ [x,C] \in \Gamma\}, \\ V_j &= \pi[0, A^{-j}]V_0. \end{aligned}$$

By construction, V_0 is invariant under the π action of Γ. Moreover, the subspaces have dense union and trivial intersection. Unfortunately, it is not necessarily the case that $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$. This condition is satisfied however, if $A \cdot R$ can be tiled by tiles from $\Gamma \cdot R$. Namely, if there exists $[x_1, C_1], \ldots, [x_d, C_d] \in \Gamma$ such that

$$A \cdot R = \bigcup_{j=1}^{d} [x_j, C_j] \cdot R.$$

It may be that there are no such group elements for a particular fundamental domain, see the example below when $\Gamma = \mathfrak{p}6$. If this is the case, a partition into subsets $R = R_1 \cup \cdots \cup R_r$, so that $A \cdot R$ may be tiled by tiles from $\Gamma \cdot R_i$, $i = 1, \ldots, r$ may be used. That is, we now desire,

$$A \cdot R = \bigcup_{j=1}^{d} ([x_{j,1}, C_{j,1}] \cdot R_1 \cup \cdots \cup [x_{j,r}, C_{j,r}] \cdot R_r)$$

for some $[x_{1,i}, C_{1,i}], \ldots, [x_{d,i}, C_d^i] \in \Gamma$, $i = 1, \ldots, r$. If this is the case, then $\{V_j\}_{j \in \mathbb{Z}}$ becomes an $\mathrm{MRA}_{A\Gamma}$ with FSE $\{\varphi_1, \ldots, \varphi_r\}$, where $\varphi_i = \chi_{R_i}/||R_i||$.

In each of the examples that follow, we will choose bases $\mathbf{a}$ and $\mathbf{b}$, so that the lattice can be represented by $\mathbb{Z}^2$.

5.1 Square Lattice : $\Gamma = \mathfrak{pg}$

Take σ_0 to be a reflection about $\mathbf{a}$. Then the subgroup $\mathfrak{pg}$ of $\mathrm{Aff}_n(\mathbb{R})$ defined by,

$$\mathfrak{pg} = \{[n, I], [\frac{\mathbf{a}}{2}, \sigma_0][m, I] \;:\; n, m \in \mathbb{Z}^2\},$$

is a wallpaper group. It is generated by integer translations $[n, I]$ and a glide reflection $[\frac{\mathbf{a}}{2}, \sigma_0]$ about the $\mathbf{a}$-axis. It is important to note that this group is *not* a semi-direct product due to the half integer translation in the glide reflection. As a result, the theory of wavelets with composite dilations as formulated in (1) from section 2 does not apply. A pattern representing $\mathfrak{pg}$ is illustrated in figure 1.

Now, take the shaded region R depicted in figure 2 as fundamental domain and

$$A = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix},$$

where $A \neq I$ and s, t are such that $A\mathbf{a}$ has odd entries. Then A is compatible with $\mathfrak{pg}$. Viz.,

$$\begin{aligned} [0, A]\,[x, B][0, A^{-1}] &= [Ax, ABA^{-1}] \\ &= \{[An, I], [\frac{1}{2}A\mathbf{a}, A\sigma_0 A^{-1}][Am, I] \;:\; n, m \in \mathbb{Z}^2\}. \end{aligned}$$

Clearly $An, Am \in \mathbb{Z}^2$. The reflection σ_0 about the $\mathbf{a}$-axis is diagonal and hence is invariant under conjugation with another diagonal. Since $A\mathbf{a}$ has odd entries, $[\frac{1}{2}A\mathbf{a}, \sigma_0]$ is a glide along the $\mathbf{a}$-axis. Therefore, $[0, A]\mathfrak{pg}[0, A^{-1}]$ is a proper subgroup of $\mathfrak{pg}$ with index $|\det A|$.

Now, the normalized characteristic function φ of R will generate an $\text{MRA}_{A,\mathfrak{pg}}$. The condition $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$ is illustrated by figure 2 for $s = 2$ and $t = 3$. The scaled tile $A \cdot R$ (outlined region) is tiled by the smaller tiles from $\mathfrak{pg} \cdot R$.

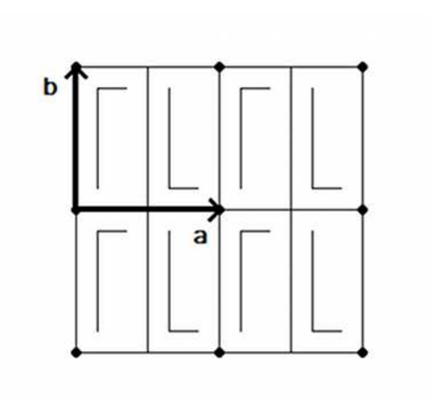

Figure 1: Pattern representing wallpaper group $\mathfrak{pg}$.

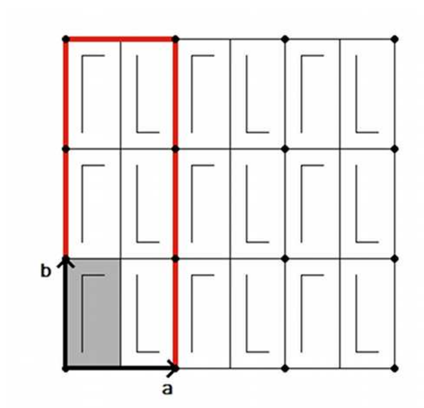

Figure 2: Fundamental domain R (shaded region). Action of A on R with $s = 2$ and $t = 3$ (outlined region).

5.2 Hexagonal Lattice : $\Gamma = \mathfrak{p}6$

Take Γ to be the wallpaper group $\mathfrak{p}6$ defined by,

$$\mathfrak{p}6 = \mathbb{Z}^2 \rtimes \{I, \rho_6, \rho_6^2, \ldots, \rho_6^5\},$$

where ρ_6 is a rotation about the origin by $\pi/3$. Set,

$$A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},$$

$A \neq I$ and $s \in \mathbb{Z}$. Clearly the lattice is preserved by A and since every element of $\mathrm{GL}_n(\mathbb{R})$ is invariant under conjugation by A the compatibility requirement is met.

If we use the shaded region R in figure 3 as our fundamental domain, then $\varphi = \chi_R/||R||$ does not generate an $\mathrm{MRA}_{A,\mathfrak{p}6}$. As illuatrated with $s = 2$, the nesting condition $V_j \subset V_{j+1}$ is not satisfied since the scaled region $A \cdot R$ cannot be tiled by tiles from $\mathfrak{p}6 \cdot R$, see figure 3.

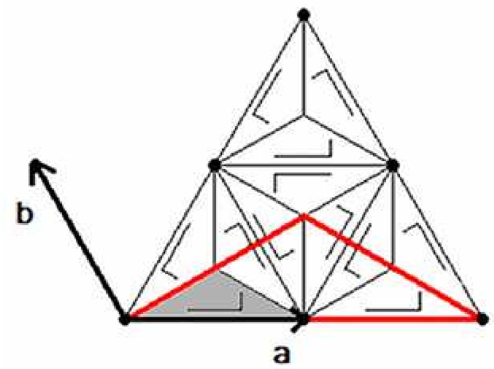

Figure 3: Pattern representing wallpaper group $\mathfrak{p6}$. Fundamental domain R (shaded region). Action of $A = \operatorname{diag}(2, 2)$ on R (outlined region).

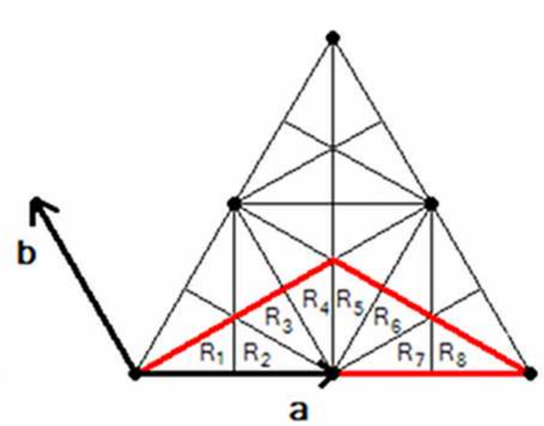

Figure 4: Decomposition of fundamental domain $R = R_1 \cup R_2$.

However, if we decompose R into $R_1 \cup R_2$, see figure 4, we do obtain an $\text{MRA}_{A,\mathfrak{p6}}$ with FSE $\{\chi_{R_1}/||R_1||, \chi_{R_2}/||R_2||\}$. The scaled region $A \cdot R$ can be tiled by tiles from $\mathfrak{p6} \cdot R_1$ and $\mathfrak{p6} \cdot R_2$, i.e. $A \cdot R = R_1 \cup \cdots \cup R_{r \cdot d}$, where $r = 2$ and $d = |\det A| = 4$.

5.3 Hexagonal Lattice : $\Gamma = \mathfrak{p6mm}$

The group $\mathfrak{p6mm}$ is the largest wallpaper group. It is the symmetry group of a tessellated hexagon,

$$\mathfrak{p6mm} = \mathbb{Z}^2 \rtimes \{I, \rho_6, \rho_6^2, \rho_6^3, \rho_6^4, \rho_6^5, \sigma_0, \rho_6\sigma_0, \rho_6^2\sigma_0, \rho_6^3\sigma_0, \rho_6^4\sigma_0, \rho_6^5\sigma_0\},$$

where ρ_6 is a rotation about the origin by $\pi/3$ and σ_0 is a reflection about an axis. Take the shaded region R in the figures below as fundamental domain. In this case, both

$$A_1 = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 2s & -s \\ s & s \end{pmatrix},$$

with $s \in \mathbb{Z}$, yield an $\mathrm{MRA}_{A,\mathsf{p6mm}}$ generated by the scaling vector $\varphi = \chi_R/||\chi_R||$. Again the lattice is preserved by each matrix and A_1 leaves the point group invariant. The action of A_2 on the point group yields the following permutation,

$$\left(\begin{array}{cccccc|cccccc} e & \rho_6 & \rho_6^2 & \rho_6^3 & \rho_6^4 & \rho_6^5 & \sigma_0 & \rho_6\sigma_0 & \rho_6^2\sigma_0 & \rho_6^3\sigma_0 & \rho_6^4\sigma_0 & \rho_6^5\sigma_0 \\ e & \rho_6 & \rho_6^2 & \rho_6^3 & \rho_6^4 & \rho_6^5 & \rho_6\sigma_0 & \rho_6^2\sigma_0 & \rho_6^3\sigma_0 & \rho_6^4\sigma_0 & \rho_6^5\sigma_0 & \sigma_0 \end{array}\right).$$

As a result, both meet the compatibility requirement.

The action of A_1 on R with $s = 2$ and $s = 3$ is indicated in figure 5. See figure 6 for the action of A_2 on R with $s = 1$ and $s = 2$.

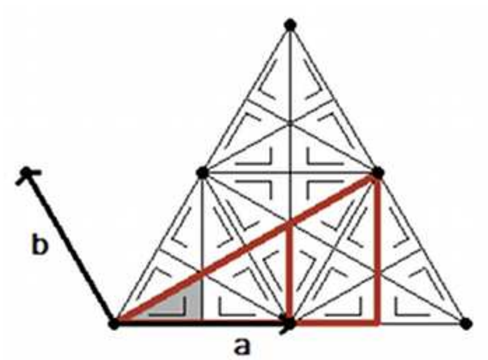

Figure 5: Pattern representing wallpaper group $\mathfrak{p6mm}$. Fundamental domain R (shaded region). Action of A_1 with $s = 2$ and $s = 3$ on R (outlined region).

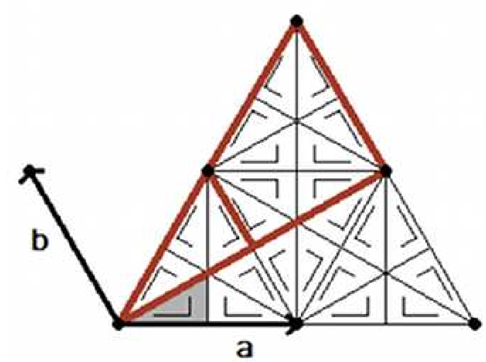

Figure 6: Action of A_2 with $s = 1$ and $s = 2$.

In each case the scaled region $A_i \cdot R$ can be tiled by tiles from $\mathfrak{p6mm} \cdot R$ satisfying the nesting condition.

In [7], examples of Haar-type wavelets associated with $\mathfrak{p6}$ and $\mathfrak{p6mm}$ are constructed within the context of composite dilations. We chose to treat these same examples from the crystal shifts viewpoint and using a basis chosen so that the dilation matrices are over the integers, to provide a comparison between the two approaches in cases where they overlap.

6 Concluding remarks

The basic ingredients needed for the development of wavelet theory when the group of shifts is non-translational and non-abelian, including definitions of multiwavelets, generalized multiresolution analysis and finite scaling ensembles, were introduced. The essential existence theorem for a multiwavelet in the presence of a GMRA with a finite scaling ensemble was established and examples illustrated Haar-type constructions for various two dimensional crystal groups as the group of shifts. The fact that nonsymmorphic crystal groups do not fit into the composite dilation framework provides the incentive to continue to develop the theory we introduced here in parallel to the fruitful line of research into composite dilations. It is our belief that the accessible representation theory of crystal symmetry groups will support the development of many of the tools for a mature wavelet analysis in this setting. Tools such as a theory of shift-invariant subspaces of $L^2(\mathbb{R}^n)$ for non-abelian shift groups acting on $\mathbb{R}^n$.

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