Solutions to Sample Problems

1. Note that in each turn, the total number of marbles in the jar decreases by exactly one. This is because we are always taking away two marbles and putting back one. So eventually we will only have one marble left. Our task is to show that this final marble is red. The key idea in this solution is to look at the *parity* of the red marbles (i.e., look at whether the number of red marbles is even or odd).

   In case (a) and (b), the jar loses one green marble, and in case (c), the jar loses two red marbles and gains one green marble. Thus, the number of red marbles in the jar must decrease by 0 or 2 on each turn. So if we have 5 red marbles, we could get down to 3 red marbles or 1 red marble, but we can't ever get down to 0. In other words, the number of red marbles must always be *odd*, because we start with an odd number of red marbles, and reducing this quantity by two does not change its parity (i.e. it's evenness/oddness).

   So at the end of the procedure, Zenia cannot remove all of the red marbles, because there must be an odd number of red marbles at all times. It follows that the green marbles must all eventually disappear, and we conclude that Zenia's final marble must be red.

2. *This problem appeared on the 2001 Canadian Mathematical Olympiad.*

   Let b be Rachel's age, and let c be Jimmy's age. Then our quadratic equation must be of the form $a(x-b)(x-c)$, since the roots of the quadratic are b and c.

   Expanding, we have $a(x-b)(x-c) = ax^2 - (ab+ac)x + abc$. The sum of the coefficients is $a - (ab+ac) + abc = a(bc - b - c + 1) = a(b-1)(c-1)$, which we are given is a prime number. First of all, b and c are both positive integers, and they must both be at least 2 (otherwise this product $a(b-1)(c-1)$ is 0, which is not prime). Furthermore, prime numbers are by definition *positive*, so $a > 0$, since $(b-1) > 0$ and $(c-1) > 0$.

   We are given that $a(b-1)(c-1)$ is a prime. Since this number is the product of three positive integers, it follows that two of the numbers must be 1. So $a = 1$ and $(c-1) = 1$, since Rachel is older than Jimmy. It follows that $c = 2$, i.e., that Jimmy is two years old.

   Our quadratic is therefore $(x-b)(x-2)$. Suppose Randy substitutes $x = k$ into the equation to get $(k-b)(k-2) = -55$. Since $b > 2$, it follows that $(k-b) < (k-2)$. There are four possible cases to consider:

$k-b$	-1	-55	-5	-11
$k-2$	55	1	11	5

   In the first two cases, we find that $b = 58$, which is impossible because $a(b-1)(c-1) = 57$, which is not prime. (Not to mention a 58 year old can't possibly have a 2 year old sibling!) And the last two cases give us $b = 18$, which make $a(b-1)(c-1) = 17$, which is prime.

   Thus, we conclude that Rachel's quadratic is $(x-18)(x-2) = x^2 - 20x + 36$, and that she is 18 years old.

3. *This problem appeared on the 1986 Canadian Mathematical Olympiad.*

There were M events and in each event, a total of $p+q+r$ points were awarded. Hence, there were $M(p+q+r)$ total points awarded. Since the total number of points was $22+9+9=40$, it follows that $M(p+q+r)=40$.

Since $p>q>r>0$, we have $p+q+r \geq 3+2+1=6$, and so there are only four cases to consider:

(a) $M=1$ and $p+q+r=40$.
(b) $M=2$ and $p+q+r=20$.
(c) $M=4$ and $p+q+r=10$.
(d) $M=5$ and $p+q+r=8$.

We can immediately reject the first case. If $M=2$, then Prima got at most $p+q$ points, because she came second in one of the two events. But $p+q<p+q+r=20<22$, so she could not have possibly gotten 22 points. So we can reject this case too.

If $M=4$, then $p+q+r=10$. The possible values for (p,q,r) are $(7,2,1)$, $(6,3,1)$, $(5,4,1)$, and $(5,3,2)$. For $(p,q,r)=(7,2,1)$, the worst Karan could have done was to come last in the rest of her events, so she must have at least $p+3r=10$ points. However, she got 9 points. For the other three cases for $M=4$, the best Prima could have done was $3p+q$, since she did not win the shot put. But for each of $(6,3,1)$, $(5,4,1)$, and $(5,3,2)$, we have $3p+q<22$. Thus, we must reject these cases as well. So $M \neq 4$.

That leaves us with $M=5$ as the only possibility. Since $p+q+r=8$, we have either $(p,q,r)=(4,3,1)$ or $(p,q,r)=(5,2,1)$. In the first case, the best Prima could have done was finish with $4p+q=19$ points, so she could not have possibly gotten 22 points. Hence, we must have $p=5, q=2, r=1$. The best Prima could have done was score $4p+q=22$ points, which is exactly what she got. So Prima must have won every event other than the shot put. The worst Karan could have done was score $p+4r=9$ points, which is exactly what she got. So Karan must have come third in every event other than the shot put. Hence, it follows that Donna must have come second in every event other than the shot put (indeed, if she finished last in the shot put and second in everything else, she got $4q+r=9$ points).

Thus, we conclude that $M=5$, and Donna finished second in the high jump.

4. *This appeared as Question 10 in Game 1 of the 2003-2004 Nova Scotia High School Math League.*

We know that $a+b+c+d+e=a+(b+c)+d+e=a+2d+e$ is a prime number. Clearly this prime number is odd (the only even prime is 2). So that means $a+e$ must be odd. What if one of a and e were not prime? Then that would force b,c,d to all be prime, while still satisfying $d=b+c$. The only way this could occur is if $b=2$ and c and d are odd primes differing by 2, such as $(b,c,d)=(2,11,13)$. But that would force $a=1$ and then e would have to be even (since $a+e$ is odd). In other words, a and e are both composite, which contradicts the fact that four of our five integers are prime.

So a and e must both be prime, and add to an odd integer. The only way this could occur is if $a = 2$, since 2 is the only even prime. Now, let's look at the final condition. We know that e is prime, and that $\sqrt{e-2}$ is some prime number p. Squaring and simplifying, we have $e = p^2+2$. If p is a prime of the form $3k+1$, then $e = (3k+1)^2+2 = (9k^2+6k+1)+2 = 3(3k^2+2k+1)$, which is not prime. If p is a prime of the form $3k+2$, then $e = (3k+2)^2+2 = (9k^2+12k+4)+2 = 3(3k^2+4k+2)$, which is not prime. So p must be a multiple of 3. Since p is prime, this implies that p must equal 3. Hence $e = p^2+2 = 11$, which is also prime.

We know that $a = 2$ and $e = 11$. Now simple case-checking shows that the only possible solutions are $(a,b,c,d,e) = (2,3,5,8,11), (2,3,7,10,11), (2,3,4,7,11)$. The last two we can eliminate because their sum is not prime. It follows that the only possible solution is $(a,b,c,d,e) = (2,3,5,8,11)$. Checking, we see that all five conditions are satisfied.

The desired answer is $2^3+3^3+5^3+8^3+11^3$, which amazingly works out to 2003. (Coincidence?)

5. Suppose that $\sqrt{3}$ is rational. In other words, suppose that we can write this number as a fraction (reduced to lowest terms). So there exist positive integers a and b, with $\gcd(a,b) = 1$ so that $\sqrt{3} = \frac{a}{b}$. We will establish a contradiction.

   Let's square both sides. Thus, $3 = \frac{a^2}{b^2}$, and we can rewrite this as $a^2 = 3b^2$. Since $3b^2$ is a multiple of 3, that implies that a^2 is also a multiple of 3. If a^2 is a multiple of 3, that must mean that a is a multiple of 3 (since 3 is prime). Hence, a is a multiple of 3, and so we can write $a = 3m$, for some positive integer m.

   Substituting into our equation, we have $3b^2 = a^2 = (3m)^2 = 9m^2$, which reduces to $b^2 = 3m^2$. Now we have the same argument as before: since $3m^2$ is a multiple of 3, that means b^2 is a multiple of 3. And this implies that b is a multiple of 3.

   So we have proven that a and b must both be multiples of 3. Then these two numbers have a common factor of 3, but this contradicts the given stipulation that $\gcd(a,b) = 1$. Therefore, we have established a contradiction, and so $\sqrt{3}$ cannot be expressed in the form $\frac{a}{b}$, where a and b are positive integers. Hence, we have proven that $\sqrt{3}$ is irrational.

6. Since x is a two-digit number, let $x = 10a+b$, where $1 \le a \le 9$ and $0 \le b \le 9$. Then $y = ab$, since it is the product of the digits of x. We are given that $x+y = 10a+b+ab = 66$.

   Solution 1: Adding 10 to both sides and factoring, we have:

$$\begin{aligned} 10a+b+ab &= 66 \\ 10a+b+ab+10 &= 76 \\ a(b+10)+(b+10) &= 76 \\ (a+1)(b+10) &= 76 \end{aligned}$$

   Since b is a digit, $b+10$ must be between 10 and 19. The only divisor of 76 in this range is 19, and so $b+10$ must be 19. Thus, $a+1$ must be 4. Solving, we get $a = 3$ and $b = 9$, and so $x = 39$. Hence, $y = 27$.

Solution 2: Solving for b in terms of a, we have:

$$\begin{aligned}
10a + b + ab &= 66 \\
b(a+1) + 10a &= 66 \\
b(a+1) &= 66 - 10a = 76 - (10a + 10) \\
b &= \frac{76}{a+1} - \frac{10a+10}{a+1} \\
b &= \frac{76}{a+1} - 10
\end{aligned}$$

Since a and b are both integers, $\frac{76}{a+1}$ must be an integer. Thus, $a+1$ must divide $76 = 2^2 \times 19$. Since a is a digit, a must be between 1 and 10. So $a+1$ is either 2 or 2^2. If $a+1 = 2$, then $a = 1$ and $b = \frac{76}{2} - 10 = 28$, which is not a digit. So the only possibility is to have $a+1 = 4$, so $a = 3$ and $b = \frac{76}{4} - 10 = 9$. Thus, $x = 39$ and $y = 27$.