Power Questions - Solutions

1. We rearrange the letters of the word “RESETS” to form new words like “RETSSE” (these don’t have to be English words). We put all these words into a list in alphabetic order, so the first word in the list is “EERSST” while the last one is “TSSREE”.

   a) How many distinct words are in this list?

   There are 6 letters in the word, so there are 6! different rearrangements possible. Since we may interchange the Ss or Es without changing the word, we must divide by 2 for each giving a total of $\frac{6!}{2 \cdot 2} = 180$ distinct words in the list.

   b) A word is selected at random from the list. What is the probability that both “S”s are next to each other?

   To determine the probability, we can simply count how many words have this property, then divide by the total number of words which was calculated in part a). To determine the number of words where both Ss are adjacent, we can consider the pair as a single letter. Then there are 5! rearrangements, but we must still divide by 2 for the repetition of the letter E. Therefore, there are $\frac{5!}{2} = 60$ words where SS occurs. Therefore, the probability of choosing one at random from the list is $\frac{60}{180} = \frac{1}{3}$.

   c) What is the 139th word in this list?

   We can start by determining what the first letter of this word must be. Using the same technique as before, we can determine that there are $\frac{5!}{2} = 60$ words that start with the letter E, $\frac{5!}{4} = 30$ words that start with the letter R, 60 that start with S and 30 that start with T. Therefore words 91 through 150 all start with the letter S.

   To determine the next letter of the word, we repeat the process with all the 5 letter words that can be made using the letters “EERST”.

There are $4! = 24$ that start with E, 12 that start with R, 12 that start with S and 12 that start with T. Therefore, words 139 through 150 all have the second letter being T. Since 139 is the first word that starts with "ST", we know the rest of the letters are in alphabetical order. Therefore, the 139th word in the list is "STEERS".

2. Angela and Paul are playing a game with Angela playing first. They start with a positive integer n. On each player's turn, they may move to a new integer by either dividing n by 2, or subtracting from n the largest power of 2 which is less than n. For example, if $n = 20$ then Angela could start by moving to 10 or 4. If a player cannot make a legal move, they lose.

   a) Find all n for which Angela has two different choices for her first move.

   In order for Angela to have two different choices it must be the case that $\frac{n}{2}$ and the largest power of 2 less than n are distinct integers. First we notice that we will require that n is even otherwise $\frac{n}{2}$ isn't an integer. Now we can ask: when is the largest power of 2 less than n equal to $\frac{n}{2}$? This occurs exactly when n is itself a power of 2. Therefore, Angela has two different choices for her first move when n is even but not a power of 2.

   b) Find all n with $1 \leq n \leq 10$ for which Angela will win the game.

   First we must make the observation that the first player will win the game if they have a legal move to a number for which the second player can win (since they will now be the second player from this point). Similarly, the second player will win the game if the first player has no such move.

   So, from $n = 1$, there are no legal moves for the first player which means the second player will win. Then for $n = 2$, the first player can move to 1 from which they will win. In this manner we can construct the following table:

n	legal moves	who wins?
1	–	Paul
2	1	Angela
3	1	Angela
4	2	Paul
5	1	Angela
6	$2, 3$	Paul
7	3	Paul
8	4	Angela
9	1	Angela
10	$2, 5$	Paul

Therefore, Angela can win the game if we start with the numbers 2,3,5,8 or 9.

c) Given a particular n, show that every sequence of play will have the same number of moves.

For any given n, we can consider its binary representation. Then we see that a legal move is to remove a leading 1 (which is equivalent to subtracting the largest power of 2 less than n) or to remove a trailing 0 (which is equivalent to dividing n by 2). Then we make the observation that for any given n, the game will only terminate when someone moves to 1. This will happen exactly when all trailing 0s and leading 1s have been removed from the binary representation. Since the binary representation is unique, we know this will take a fixed number of moves for any given n.

3. In the following triangle of numbers, each entry is the sum of the three numbers above it in the previous row. For instance, in the 5th row, the number 16 is determined by the sum of the numbers 3, 6 and 7 in the previous row.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

a) Explain why the middle entry of every row is odd.

We can see that initially, the middle position is odd. For each later row, we know that by symmetry the value one space to the left and right of centre will be the same. Therefore, the middle three values will be even-odd-even or odd-odd-odd. In either case, their sum must be odd and therefore, the middle term in the next row will be odd. This allows us to conclude that the middle number in each row will always be odd.

b) Show that every row beyond the second contains at least one number which is even.

Assume there is a row beyond the second for which every number is odd. Then, using "o" to represent odd and "e" to represent even we can work backwards to get the following

$$
\begin{array}{ccccccccccccccccccc}
 & & 1 & o & e & e & e & e & o & o & e & e & e & e & o & o & e & e & \ldots \\
 & 1 & e & e & o & e & e & o & e & e & o & e & e & o & e & e & o & e & \ldots \\
1 & o & o & o & o & o & o & o & o & o & o & o & o & o & o & o & o & o & \ldots
\end{array}
$$

This demonstrates that the row before the one which is all odd follows the pattern odd-even-even indefinitely. Likewise, the row previous to that follows the pattern odd-odd-even-even-even-even. (Note: the existence of the these patterns can be proven rigorously but is not required for a full solution).

Now we can see that it will never be the case that there is any column for which there is a odd number in all three rows. This does not make sense since in part a) we concluded that the middle of each row was odd. Therefore, our original assumption must have been false and there can be no row beyond the second which only contains odd numbers. This is equivalent to saying that every row beyond the second has at least one even number in it.