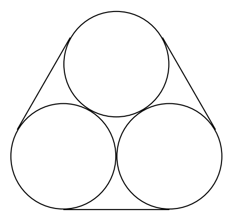
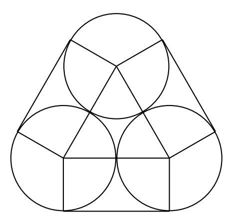
Set 4 - Solutions

1. Three pipes with radius 1 are bound together by an iron band. What is the length of the band?



To begin, we make the following constructions:



Then we notice that each of the rectangles is 2 units long. Therefore, the straight portions of the band must have a total length of 6. We also note that the arc sections are each exactly 120 degrees. Since there are three of them, their sum will be the same as one full circle. Since these circles have radius 1, their circumference is 2π. So, the total length of the band is $6 + 2\pi$ or 12.2832.

2. A parabola with integer roots passes through the points (1,-6) and (6,44). Find the equation of this parabola.

   We know that the equation of a parabola with integer roots can be expressed as $y = (x - s)(x - t)$ where s and t are the roots. Using the points given in the problem, we know that $-6 = (1 - s)(1 - t) = 1 - (s + t) + st$ and $44 = (6 - s)(6 - t) = 36 - 6(s + t) + st$. This gives us the system:

$$\begin{aligned} 8 &= -6(s+t) + st \\ -7 &= -(s+t) + st \end{aligned}$$

   and hence we can calculate:

$$\begin{aligned} 15 &= -5(s+t) \\ 3 &= -(s+t) \end{aligned}$$

   and by substitution we find

$$\begin{aligned} -7 &= 3 + st \\ -10 &= st \end{aligned}$$

   Therefore, the equation of the parabola must be $y = x^2 + 3x - 10$.

3. Find the smallest positive integer n such that $2n$ is a perfect square, $3n$ is a perfect cube, $5n$ is a perfect fifth power.

   To find the smallest number with the given properties, we note that we only need to have prime factors of 2, 3 and 5. To determine the number of 2s required, we note that since 2n is a perfect square, we need an odd number of 2s. Also, since 3n is a perfect cube we need the number of 2s to be a multiple of 3. Likewise, for $5n$ to be a perfect fifth power, we need the number of 2s to be a multiple of 5. The smallest number which satisfies these conditions is 15. Therefore, n must have a factor of 2^{15}. We continue in the same manner to determine that the number of 3s needed must be even, a multiple of 5 and one less than a multiple of 3. The smallest choice is therefore 20 which gives

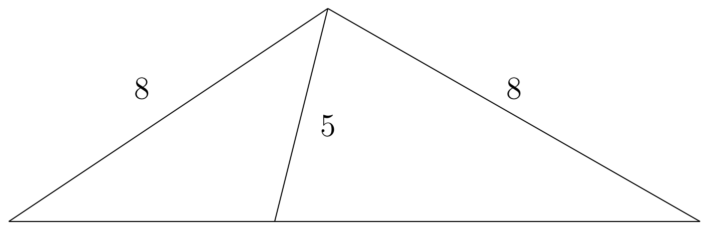
n a factor of 3^{20}. Finally, we find the smallest number of 5s needed. There must be a multiple of 6 (2 and 3) of them, and one less than a multiple of 5. A quick search tells us that 24 is the smallest such number. Therefore, n must have a factor of 5^{24}. Since n is to be as small as possible we can conclude that these are the only factors of n and therefore $n = 2^{15}3^{20}5^{24} = 6810125783203125000000000000000$.

4. We say that a four-digit integer $ABCD$ is *amiable* if all four digits are different, and $ABCD$ is divisible by each of A, AB, and ABC. For example, 1260 is amiable because all four digits are different, and 1260 is divisible by each of 1, 12, and 126. Find the largest amiable number.

   We can start by determining the value of D. Since $ABCD$ is divisible by ABC we can conclude that $D = 0$. Now, since AB divides $ABCD$ we know that AB must divide CD. Since CD is a multiple of 10, to make AB as large as possible it would have to be a divisor which doesn't repeat the digit 0. The largest possible divisor would therefore occur when C was odd and $2 \cdot AB = CD$ which would make $B = 5$. The largest possible choice for C would therefore be 9 giving us the number 4590. Unfortunately, this number is not divisible by 4 and hence not amiable. The next largest candidate would be 3570 which is indeed divisible by 3. Therefore, this is the largest amiable number.

5. Find the sum of the areas of the distinct triangles ABC which can be formed where $|AB| = 5$, $|BC| = 8$ and $\angle BCA = 30°$.

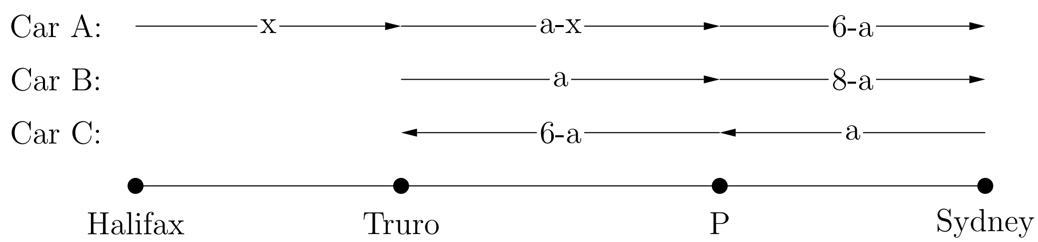
   First of all, we need to recognize that there are two distinct triangles which can be constructed using the information given. Also, if we place these two triangles next to each other so that their sides of length 5 are next to each other we get the following diagram:



Therefore, the sum of their areas is simply the area of this isosceles triangle. Since the small angle is 30 degrees, we can recognize that this is similar to two $1-2-\sqrt{3}$ triangles. Therefore, the base is $8\sqrt{3}$ and the height is 4. We then calculate that the area is $16\sqrt{3} = 27.7128$.

6. Car A leaves Halifax and travels to Sydney (passing through Truro) at a constant speed so that the entire trip will take 6 hours. Car B leaves Truro and travels to Sydney at a constant speed so that their trip will take 8 hours. Car C leaves Sydney and travels to Truro at a constant speed so that the trip will take 6 hours. All the cars leave at the same time and they all meet at the same point along the route. How long did it take Car A to get from Halifax to Truro?

   To begin, we will draw a diagram showing the time it takes each car takes to make its trip. We will label the point P as where all three cars meet. We will also say that it takes a hours for the cars to reach point P and that it takes x hours for car A to get from Halifax to Truro. So, we get the following diagram:



First we consider the trips taken by cars B and C. Since they have travelled the same distance at constant rates we can calculate:

$$\begin{aligned}
\frac{a}{6-a} &= \frac{8-a}{a} \\
a^2 &= 48 - 14a + a^2 \\
a &= \frac{48}{14} \\
a &= \frac{24}{7}
\end{aligned}$$

Now, using the same technique with cars A and C we get

$$\begin{aligned}
\frac{a-x}{6-a} &= \frac{6-a}{a} \\
a^2 - ax &= 36 - 12a + a^2 \\
ax &= 12a - 36 \\
x &= 12 - \frac{36}{a} \\
x &= 12 - 36\frac{7}{24} \\
x &= 12 - 10.5 \\
x &= 1.5
\end{aligned}$$

Therefore, it took car A 1.5 hours (or 90 minutes) to travel from Halifax to Truro.

Relay Solution Key:
Relay 'A':
$A = 3$
$B = 8$
$C = 17$
$D = -\frac{1}{17}$

Relay 'B':
$A = 1$
$B = 6$
$C = 9$
$D = 27$