Set 4 - Solutions

1. Find the largest positive integer n which satisfies the following conditions:

   (a) The digits of n are all distinct.

   (b) n does not contain the digit 0.

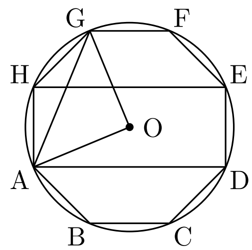
   (c) n is divisible by the sum of its digits.

   To begin, since we would like to have n as large as possible but the digits of n have to be distinct, we know that $n \leq 987654321$. If n contains all the digits from 1 to 9, it will have to be divisible by $1+2+3+4+5+6+7+8+9 = 45 = 9 \cdot 5$. But, since the sum of the digits is already divisible by 9, we only care about making n divisible by 5. Since we cannot use 0 as the last digit, we'll have to use 5. Putting the other digits in decreasing order we get $n = 987643215$.

2. Richard has \$15.25 worth of nickels and dimes in his piggy bank. If he has an odd number of dimes and he has more dimes than nickels, what is the largest number of nickels that he could have?

   If we let d be the number of dimes and n be the number of nickels we get the relation $10d + 5n = 1525$ which can be simplified to $2d + n = 305$. Since we know that he has more dimes than nickels we get the inequality $2d + d > 305$ which gives us $d > \frac{305}{3}$. Since d has to be an integer, we know $d > 101$. Since d has to be odd, we know $d \geq 103$. Now, to make the number of nickels as large as possible, we have to make the number of dimes as small as possible, so we pick $d = 103$. Then we get that $206 + n = 305$ and we conclude that the the maximum value for n is 99.

3. ABCDEFGH is a regular octagon with side length 1 inscribed in a circle with centre O. What is the area of the triangle AOG?



First, we note that the triangle AOG is right angled at O. Then we see that AO is a radius and the area of the triangle will be $\frac{r^2}{2}$.

Now we consider the rectangle $AHED$. The diagonals of this rectangle pass through the centre of the circle. Therefore, they are diameters of the circle. The short side length of the rectangle is 1 while the long side has length $\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$. So, using Pythagorean theorem, we know that

$$\begin{aligned} (2r)^2 &= 1^2 + (1+\sqrt{2})^2 \\ 4r^2 &= 1 + 1 + 2\sqrt{2} + 2 \\ 4r^2 &= 4 + 2\sqrt{2} \\ \frac{r^2}{2} &= \frac{2+\sqrt{2}}{4} \end{aligned}$$

Therefore, we know the area of the triangle is $\frac{2+\sqrt{2}}{4}$.

4. A positive integer is a palindrome if it reads the same forward and backwards such as 1287821 and 4554. Determine the number of 5-digit positive integers which are NOT palindromes.

We start by counting the total number of 5 digit positive integers. The first digit is between 1 and 9, so we have 9 choices. Each of the other 4 digits can be anything at all (10 choices for each). This gives us $9(10)^4 = 90000$ five-digit positive integers.

Now we need to count the number of 5 digit palindromes. Again, we have 9 choices for the first digit and 10 choices for each of the next two. The tens and units digits however are fixed by our choices so far. Therefore, there are only 900 five-digit palindromes.

Therefore, the total number of five-digit positive integers which are not palindromes is 90000-900 = 89100.

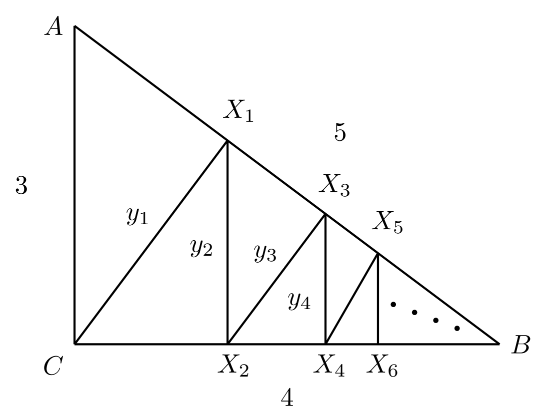
5. On an analog clock, the hour hand and the minute hand form a 90° angle. How long will it be before they form a 90° angle again?

   Since a minute hand travels a full circle every 60 minutes, we know that it moves at the rate of 6 degrees per minute. The hour hand only moves 30 degrees every 60 minutes and so is travelling at the rate of 0.5 degrees every minute. Since both hands travel in the same direction around the clock, the angle from the minute hand to the hour hand is always decreasing at the rate of $(6 - 0.5) = 5.5$ degrees every minute.

   Since the hands currently form a 90 degree angle, and we're looking for how long it will take for them to form another 90 degree angle, we can calculate the time as: $\frac{180}{5.5} = 32\frac{8}{11}$ minutes. For those that care to work it out, this is approximately 32 minutes, 43.6 seconds.

6. In the following diagram, triangle ABC has side lengths of 3, 4 and 5. Starting from C, we construct an altitude of the triangle, meeting AB at X_1. This altitude has length y_1 Now, starting at X_1 we construct an altitude of the triangle X_1BC meeting BC at X_2. This altitude has length y_2. We continue to construct altitudes in a similar way indefinitely with the length of X_nX_{n+1} being y_{n+1}. Calculate the value of

$$S = y_1 + y_2 + y_3 + y_4 + ...$$



   The first thing to note is that the triangle ABC is similar to the triangle CBX_1 because they are both right angled and contain the angle at B. Since the ratio of the hypotenuses is 5:4, we can calculate that $y_1 = \frac{12}{5}$.

When we compare the triangle X_iBX_{i-1} to the triangle $X_{i+1}BX_i$ we see that the the triangles are similar and that the longer leg of the larger triangle is the hypotenuse of the smaller triangle. Therefore, the ratio of the larger triangle to the smaller one is $4:5$ for all i.

In particular, that tells us that $y_{i+1} = \frac{4}{5}y_i$. So, when we consider the sum S, we know it is a geometric sequence with initial term $a = \frac{12}{5}$ and common ratio $\frac{4}{5}$. Therefore, the sum is:

$$\begin{aligned} S &= (a)\left(\frac{1}{1-r}\right) \\ &= \left(\frac{12}{5}\right)\left(\frac{1}{1-\frac{4}{5}}\right) \\ &= \left(\frac{12}{5}\right)(5) \\ &= 12 \end{aligned}$$

Relay Solution Key:
Relay 'A':
A = 16
B = 96
C = 480
D = 1560

Relay 'B':
A = 4
B = -2
C = -4
D = $\frac{1}{3}$