1. (a) Suppose $m$ is an integer whose only digits are zero or one. Show that if $m$ is a palindrome, then $2m, 3m, \cdots, 8m$, $9m$ are also palindromes.

   (b) Suppose $n$ is an integer such that $n$, $2n$, $3n$, $\cdots$, $8n$, $9n$ are all palindromes.

      (i) Show that the one's digit of $n$ is 1.

      (ii) Show that all digits in $n$ are either zeroes or ones.

**Proof**:

(a) Suppose $m$ is an integer whose only digits are zero or one and $m$ is a palindrome. Then we can write $m$ as $a_1a_2a_3\cdots a_3a_2a_1$ where $a_i \in \{0, 1\}$ for each $i$. Let $w \in \{2, 3, \cdots, 9\}$. We want to consider $wm$. Look at the one's digit of $wm$. Since $a_1 \in \{0, 1\}$, we have that the one's digit of $wm$ is $wa_1 \in \{0, w\}$, and since $a_1 \in \{0, 1\}$, there is no carrying. Examine the ten's digit of $wm$. Since there is no carrying, we can apply the same argument as for the one's digit to get that the ten's digit of $wm$ is $wa_2$ and there is no carrying to affect the hundred's digit. But this argument holds for any digit in $wm$, and so we can write $wm$ as $(wa_1)(wa_2)\cdots(wa_2)(wa_1)$, and this is a palindrome since $a_1a_2\cdots a_2a_1$ is a palindrome.

(b) (i) Consider the 1s digit on $n$ - call it $a$. Then, since $n$ is a palindrome, we can write $n$ as $aXa$ for some string $X$ of digits.
Suppose $a$ is even. Then the one's digit of $5a$ is 0, and so $5a$ is not a palindrome. Therefore $a$ is odd.
Suppose $a = 3$, so $n$ is $3X3$. Then $4n$ is of the form $1Y4$, contradicting that $4n$ is a palindrome.
Suppose $a = 5, 7$, or 9. Then the ones digit of $2n$ is even, while the leading digit of $2n$ is 1, contradicting that $2n$ is a palindrome.
Therefore the only value left for $a$ is 1. That is, the one's digit of $a$ is 1.

(ii) Suppose $n$ has a digit which is neither a zero nor a one. Let $t$ be the left most non-zero, non-one digit, i.e. we can write $n$ as $a_1a_2\cdots a_mt\cdots$ where $a_1, a_2, \cdots, a_m \in \{0, 1\}$ and $t \notin \{0, 1\}$. We will consider $9n$.

- Suppose $a_m = 0$. Then $9a_m = 0$. Since $t$ does not equal 0 or 1, when multiplied by nine, we will carry a digit to the next column. That is, in $9n$, the $m^{th}$ digit from the left cannot be 0 - it will be a number between 1 and 8.
- Suppose $a_m = 1$. Then $9a_m = 9$. Since $t$ does not equal 0 or 1, when multiplied by nine, we will carry a digit to the next column. That is, in $9n$, then $m^{th}$ digit from the left cannot be 1 - it will be a number in the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$.

However, since $n$ is a palindrome, we can write $n$ as $a_1a_2\cdots a_mtXta_m\cdots a_2a_1$ for some string of digits $X$. Then, considering $9n$, the $m^{th}$ digit from the right is $9a_m$. This must match the $m^{th}$ digit from the left since $9n$ is a palindrome, but we just showed above that the $m^{th}$ digit from the left is not $9a_m$.
Thus we cannot find such a $t$ and so all digits of $n$ must be zeroes or ones.

■

2. Two positive integers $m$ and $n$ are **coprime** if their greatest common factor is 1. For example, 2 and 7 are coprime. Let $\phi(n)$ be the number of positive integers less than $n$ which are coprime to $n$. For example, $\phi(6) = 2$ since the only positive integers less than 6 which are coprime to 6 are 1 and 5.

   (a) Suppose $p$ prime. Determine $\phi(p)$ and explain your answer.

   (b) Suppose $p$ prime and $k$ a positive integer. Determine $\phi(p^k)$ and explain your answer.

   (c) Suppose $p, q$ prime. Show that $\phi(pq) = (p-1)(q-1)$.

   **Proof**:

   (a) Since $p$ is prime, the only divisors of $p$ are 1 and $p$. So every positive integer less than $p$ is coprime to $p$. That is, $\phi(p) = p - 1$.

   (b) $\varphi(p^k) = p^{k-1}(p-1) = p^k - p^{k-1}$.

   There are $p^k - 1$ positive integers less than $p^k$.

   Rather than determine which numbers are coprime to $p^k$, we will determine which numbers are not coprime to $p^k$, i.e. which positive integer $t < p^k$ such that $\gcd(p^k, t) > 1$, and subtract these from the total number of positive integers less than $p^k$.

   Since $p$ prime, the only positive integers which are not coprime to $p^k$ and less than $p^k$ are integers which are of the form $ap$ for some $a \in \{1, 2, \cdots, p^{k-1} - 1\}$. That is, there are $p^{k-1} - 1$ such positive integers. Therefore there are $p^k - 1 - (p^{k-1} - 1)$ positive integers less than $p^k$ which are coprime to $p^k$. That is

$$\begin{aligned}\phi(p^k) &= p^k - 1 - (p^{k-1} - 1) \\ &= p^k - 1 - p^{k-1} + 1 \\ &= p^{k-1}(p-1)\end{aligned}$$

   (c) There are $pq - 1$ positive integers less than $pq$.

   Rather than determine which numbers are coprime to $pq$, we will try to determine which numbers are not coprime to $pq$, i.e. which positive integers $t < pq$ such that $\gcd(pq, t) > 1$ and subtract these from the total number of positive integers less than $pq$.

   Since $p$ and $q$ are prime, the prime divisors of $pq$ are $p$ and $q$, and so the only positive integers $t < pq$ such that $\gcd(pq, t) > 1$ must have either $p$ or $q$ as factors. Therefore the possibilities are, recalling that $0 < t < pq$, $p, 2p, \cdots, (q-1)p$ and $q, 2q, \cdots, (p-1)q$. Therefore, there are $p + q - 2$ positive integers less than $pq$ which are not coprime to $pq$. That is, there are $pq - 1 - (p + q - 2)$ positive integers less than $pq$ which are coprime to $pq$. Therefore

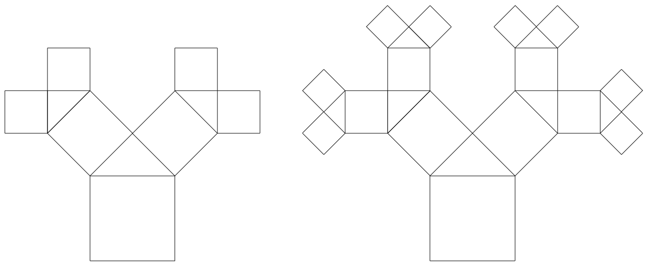
$$\phi(pq) = pq - 1 - (p + q - 2)$$

$$= pq - p - q + 1$$
$$= (p-1)(q-1)$$

■

3. The tree *arborus fractalus* grows as follows:

- Year One: A square trunk of side length eight is grown.
- Year Two: An isosceles right-angled triangle grows on top of the square with the hypotenuse of the triangle the top side of the square. Two square branches grow from the legs of the triangle.
- For subsequent years, the pattern repeats, i.e. isosceles triangles grow out of the top of each branch and square branches grow from the legs of the triangle.




(a) After $n$ years, find the ratio of squares to triangles in the tree. Show how you obtained your answer.

(b) After $n$ years, is the tree taller than it is wide or wider than it is tall? Explain your answer.

(c) Suppose $n$ even. Show that the height of the tree after $n$ years is $32 - 2^{\frac{10-n}{2}}$.

**Proof**:

(a) $(2^n - 1) : (2^{n-1} - 1)$ or $\dfrac{2^n - 1}{2^{n-1} - 1}$

Examine the number of squares we add each year. The first year, we add one square. The second year, we add two. The third year we add 4. The fourth year we add 8. We can see through the self-similarity (or prove by induction, although this is not required for this competition), that at year $n$, we add $2^{n-1}$ squares. Therefore, after year $n$, there are $1 + 2 + 4 + \cdots + 2^{n-1}$ squares. This is a finite geometric series with common ratio 2 and $n$ terms. Using the formula for the sum of an geometric series:

$$1 + 2 + 4 + \cdots + 2^{n-1} = \frac{1(1 - 2^n)}{1 - 2}$$
$$= 2^n - 1$$

Therefore after $n$ years, there are $2^n - 1$ squares.

Examine the number of triangles we add each year. In year 1, we add zero. In year two, we add one. In year three, we add 4. In year four, we add 4. We can see through the self-similarity (or prove by induction, although this is not required for this competition), that at year $n$, we add $2^{n-2}$ triangles. Therefore, after year $n$, we have $1 + 2 + 4 + \cdots + 2^{n-2}$ triangles. This is a finite geometric series with common ratio 2 and $n-1$ terms. Using the formula for the sum of geometric series:

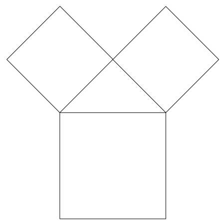
$$\begin{aligned} 1 + 2 + 4 + \cdots + 2^{n-2} &= \frac{1(1-2^{n-1})}{1-2} \\ &= 2^{n-1} - 1 \end{aligned}$$

Therefore after $n$ years, there are $2^{n-1} - 1$ triangles.

Therefore the ratio of squares to triangles after year $n$ is $2^n - 1 : 2^{n-1} - 1$ or $\dfrac{2^n - 1}{2^{n-1} - 1}$

(b) For $m \geq 3$, the tree is wider than it is tall. For $m = 1, 2$, the width and the height are the same. For full marks, both these cases need to be explicitly listed,

Clearly, after one year, the tree is eight units wide and eight units tall.

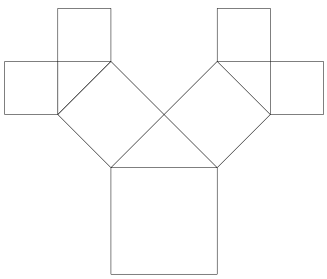


After two years, the tree is 16 units tall. This can be calculated as follows: The added height is twice the height of the triangle, since two triangles make up the new square. Let $t$ be the side length of the triangle. Then $2t^2 = 8^2$, or $t = 4\sqrt{2}$. Let $h$ be the height of the triangle. Then $h^2 + \frac{1}{2}8^2 = (4\sqrt{2})^2$. Solving for $h$ gives that $h = 4$. Therefore the height added is 8, so the height after two years is 16.

What is the width?

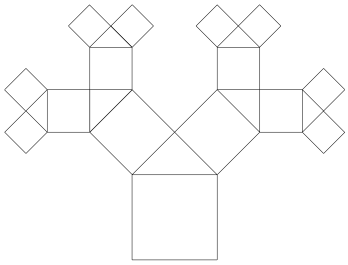
Since the height of one of the year 2 branches is 8, so is the width of one of the year 2 branches. Therefore the width is 16 - the width of the two branches. Thus the height and the width are the same.

Now suppose $m = 3$.



We see that while we add one square to the height, we add two squares to the width, so for $m = 3$, the tree is wider than it is tall.

Suppose $m = 4$.



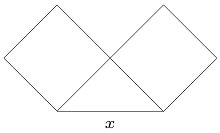
after 4 years

Note that while we are adding the height of one square, we are adding the width of two squares, and since the width after year three is already more than the height, again the tree is wider than it is tall after four years.

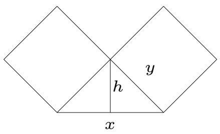
Because of the self-similarity, we can see that this trend continues, namely that each year we increase the height by some value $x$ while we increase the width by $2x$. Therefore for all $m \geq 3$, the tree is wider than it is tall.

(c) What is the height after $n$ even years?

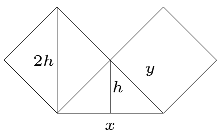
On an even year, we are adding




to the height, where $x$ is the length of the side of the square branch of the previous year. Let $y$ be the sides of the isosceles right angle triangle. By Pythagorean theorem, we have $2y^2 = x^2$ or $y^2 = \dfrac{x^2}{2}$. Let $h$ be the height of the triangle in the above diagram.




Notice that the height of the square is twice the height of the triangle since the triangle is a right angle triangle and the square grows out of the non-hypotenuse isosceles sides.



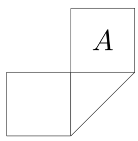

Therefore we are adding $2h$ to the height of the tree. But

$$h^2 + \left(\frac{x}{2}\right)^2 = y^2$$

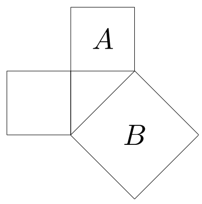
$$
\begin{aligned}
h^2 &= -\left(\frac{x}{2}\right)^2 + y^2 \\
h^2 &= -\left(\frac{x}{2}\right)^2 + \frac{x^2}{2} \\
h^2 &= -\frac{x^2}{4} + \frac{x}{2} \\
h^2 &= \frac{x^2}{4} \\
h &= \frac{x}{2}
\end{aligned}
$$

Therefore, each even year, we increase the height of the tree by $x$, the length of the side of the square from the previous year.

On an odd year, we are adding




where the only thing added to the height is the square $A$. Drawing in the square from the previous year




Then the height of the triangle is one half the height of $B$, and the height of $A$ is the height of the triangle, so the height of $A$ is one half the height of $B$. Therefore in odd years, we add to the height one half the height of the previous year's square, which, by above, is the height added the previous year.

Therefore we have shown the general case. Examining for our specific case, we have

| year | height added |
|---:|---:|
| 1 | 8 |
| 2 | 8 |
| 3 | 4 |
| 4 | 4 |
| 5 | 2 |
| 6 | 2 |
| ⋮ | ⋮ |

Because of the self-similarity of the tree, we can see that this pattern will continue. Namely, that at years $2m-1, 2m$, we add $2^{4-m}$ to the tree (this can be shown by induction, although not required, or just by referencing the self-similarity repeating structure of the tree). Therefore, the height of the tree after $n$ years is

$$
\begin{aligned}
& 8+8+4+4+2+2+\cdots+2^{4-\frac{n}{2}}+2^{4-\frac{n}{2}} \\
&= 2\cdot 8+2\cdot 4+2\cdot 2+\cdots+2\cdot 2^{4-\frac{n}{2}} \\
&= 2(8+4+2+\cdots+2^{4-\frac{n}{2}}) \\
&= 2\left(\frac{8\left(1-\left(\frac{1}{2}\right)^{\frac{n}{2}}\right)}{1-\frac{1}{2}}\right) && \text{sum of geometric series} \\
&= 32\left(1-\left(\frac{1}{2}\right)^{\frac{n}{2}}\right) \\
&= 32\left(1-\frac{1}{2^{\frac{n}{2}}}\right) \\
&= 32-\frac{2^5}{2^{\frac{n}{2}}} \\
&= 32-2^{5-\frac{n}{2}} \\
&= 32-2^{\frac{10-n}{2}}
\end{aligned}
$$

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