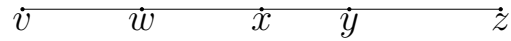


Individual Relay

Instructions

- This is the second relay.
- In this relay each student will have a question to complete. Before the relay begins, the students determine which of them will do questions 1,2,3, and 4. During the relay, there is no communication between any of the students other than passing the sheet to the next student. If a team violates this, they will receive 0 on the relay.
- The relay is ten minutes long. When the relay starts, each team's proctor distributes all the questions.
- When the fourth question has been completed, the last student hands the sheet to the proctor. If all the answers are correct, the proctor yells out the team number. If any one of the answers is incorrect, the proctor wordlessly hands the sheet back to the student who handed him the sheet. He makes no indication regarding which answer is incorrect.
- The first team to finish receives 10 points. The second team to finish receives 9 points. The third team to finish receives 8 points. The fourth team to finish receives 7 points. The fifth team to finish receives 6 points. All other teams which finish after the first five teams but within the ten minutes receive 5 points.
- If a team does not finish within the ten minutes, the proctor collects their answer sheet and brings it to the front for marking. Each correct answer is worth one mark.

- A. In the below diagram (not drawn to scale), vz has length 5, w is the midpoint of vx , x is the midpoint of wy , and y is the midpoint of wz . Let A be the length of yz .



Pass on A .

- B. You will receive A .

A points b_1, b_2, \dots, b_A are evenly placed in this order around a circle with center O . A straight line is drawn connecting O and b_1 and a straight line is drawn connecting O and b_2 . Let B be the measure in degrees of $\angle b_1Ob_2$.

Pass on B .

- C. You will receive B .

C is the unit digit of $3^B + 7^B$.

Pass on C .

- D. You will receive C .

A parabola $f(x)$ with a vertical axis of symmetry has vertex $(0, 8)$ and one of its x intercepts at $(C, 0)$. Let $D = f(5)$.

Pass on D

Answers

- A. 2
- B. 180
- C. 2
- D. -42

Full solutions

A. Let $|cd|$ denote the length of line segment cd . Let μ be the length of vw , i.e. $|vw| = \mu$. Then since w is the midpoint of vx , we have $|wx| = \mu$. Since x is the midpoint of wy , we have that $|wx| = |xy|$, so $|xy| = \mu$. We have that y is the midpoint of wz . That is $|wy| = |yz|$. But $|wy| = |wx| + |xy| = \mu + \mu$. Therefore $|yz| = 2\mu$. Then

$$5 = |vz| = |vw| + |wx| + |xy| + |yz| = \mu + \mu + \mu + 2\mu$$

we have that $\mu = 1$ so $|yz| = 2\mu = 2$.

B. Connect O to b_1, b_2, \dots, b_A . Then we have divided the circle into A equal sized slices with $\angle b_i O b_{i+1} = \frac{360}{A}$. Therefore $\angle b_1 O b_2 = \frac{360}{A}$. Since $A = 2, B = 180$.

C. Look at the unit digit of 7^n :

n	17^n
1	7
2	9
3	3
4	1
5	7
6	9
7	3
8	1
9	7
10	9
\vdots	\vdots

We can see that the pattern continues. That is

$$\text{unit digit of } 17^n = \begin{cases} 7 & \text{if } n = 4k + 1 \\ 9 & \text{if } n = 4k + 2 \\ 3 & \text{if } n = 4k + 3 \\ 1 & \text{if } n = 4k \end{cases}$$

for some $k \in \mathbb{N}$.

Now examine unit digit of 3^n :

n	3^n
1	3
2	9
3	7
4	1
5	3
6	9
7	7
8	1
9	3
10	9
\vdots	\vdots

We can see that the pattern continues. That is

$$\text{unit digit of } 3^n = \begin{cases} 3 & \text{if } n = 4k + 1 \\ 9 & \text{if } n = 4k + 2 \\ 7 & \text{if } n = 4k + 3 \\ 1 & \text{if } n = 4k \end{cases}$$

for some $k \in \mathbb{N}$.

Therefore

$$\text{unit digit of } 3^n + 7^n = \begin{cases} 0 & \text{if } n = 4k + 1 \\ 8 & \text{if } n = 4k + 2 \\ 0 & \text{if } n = 4k + 3 \\ 2 & \text{if } n = 4k \end{cases}$$

Since $B = 180 = 4 \cdot 45$, we have $C = 2$.

- D. Since the parabola has x intercepts, it is a downwards facing parabola. Since the vertex is $(0, 8)$ and the parabola has a vertical axis of symmetry, this means that the axis of symmetry is about the y axis. Thus, if one of the x intercepts is at $(C, 0)$, then the other x intercept is at $(-C, 0)$. Since the vertex is at $x = 0$, we have that $C \neq 0$.

Recall that we can write a parabola with vertical axis of symmetry in terms of the x -intercepts, namely $f(x) = a(x - C)(x + C)$ for some $a \in \mathbb{R}$. That is $f(x) = ax^2 - aC^2$. Since $(0, 8)$ is a point on the parabola, we have

$$\begin{aligned} 8 &= a0^2 - aC^2 \\ a &= \frac{-8}{C^2} \end{aligned}$$

Therefore $f(x) = \frac{-8}{C^2}(x - C)(x + C)$. Since $C = 2$, we have $f(x) = -2(x - 2)(x + 2)$. When $x = 5$, we have $f(5) = -2(5 - 2)(5 + 2) = -2(3)(7) = -42$.