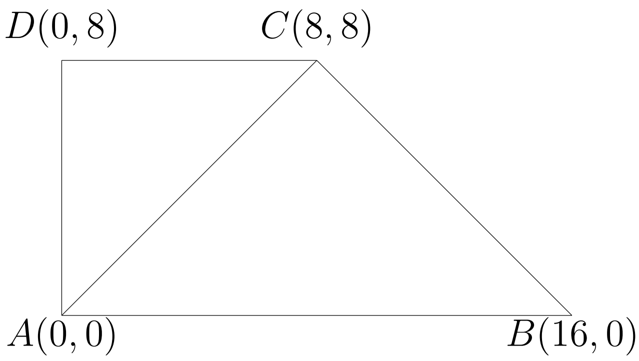
Questions

Instructions

- Before the contest begins each team should be assigned a proctor. A team's proctor cannot be their accompanying adult for the sake of impartiality.
- Each question is worth five points with a team scoring either 0 or 5 points on the question (no part marks).
- Five minutes are given to work on each question. At the beginning of the five minutes, the proctor gives each team a copy of the question. At the end of the five minutes, the answer sheet is given to the proctor and the proctor takes it to the front of the room for marking. Students must give up their sheet when time is called. Refusal to do so will result in a mark of zero for the question.
- After handing in their team's response, the proctor then picks up the next question.
- Rounded answers must be correct to three decimal places.

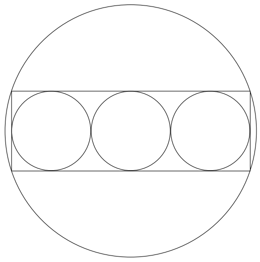
1) Suppose f is a function such that $f(x)f(y) = f(x + y) + f(x) + f(y)$ for all real x and y. Suppose also $f(0) = 3$. Find $f(2007)$.

2) A quadrilateral is drawn in the Cartesian plane with the following coordinates: $A = (0, 0)$, $B = (16, 0)$, $C = (8, 8)$, and $D = (0, 8)$. A line is drawn connecting A and C. Find the equation of the line parallel to AC which halves the area of the quadrilateral.



3) A perfectly cylindrical glass with radius 24 cm and height 50 cm is filled with water. Water is then poured from the glass until the glass is two thirds empty. A solid metal cube is then dropped into the glass and the water level rises 3 cm. What is the side length of the cube?

4) Three small circles each have diameter 6. These circles are drawn tangent to each other so their centers form a straight line. They are then inscribed in a rectangle which is inscribed in another larger circle such that the center of the large circle and the center of the small middle circle are the same. What is the area of the large circle (diagram not drawn to scale)?



5) Alice, Bridget, and Catherine roll a die in that order. Alice wins if she rolls a 1,2, or 3. Bridget wins if she rolls a 4 or 5. Catherine wins if she rolls a 6. If no one wins, the die is returned to Alice and the pattern repeats (i.e. Alice, Bridget, Catherine, Alice, etc.). What is the probability that Catherine wins?

6) When expanded, the n^{th} (i.e. 1st, 2nd, 3rd, etc.) term of

$$(a + bx + cx^2 + dx^3 + ex^4)(1 - x)^{-5}$$

is n^4x^{n-1}. Find $a + b + c + d + e$.

Answers

1) 3

2) $y = x + 8\sqrt{3} - 16$, rounded $y = x - 2.144$

3) $(1728\pi)^{\frac{1}{3}} = 12\sqrt[3]{\pi} \sim 17.575$

4) $90\pi \sim 282.743$

5) $\frac{1}{13} \sim 0.077$

6) 24

Solutions

1) Suppose f is a function such that $f(x)f(y) = f(x+y) + f(x) + f(y)$ for all real x and y with $f(0) = 3$. Find $f(2007)$.

Solution: 3

Proof: Let $x = 0$ and $y = 2007$. Then

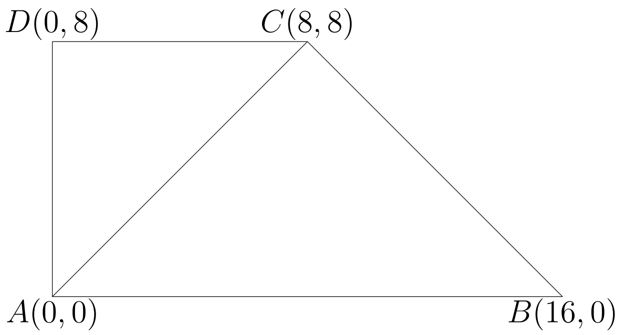
$$\begin{aligned} f(0)f(2007) &= f(0+2007) + f(0) + f(2007) \\ 3f(2007) &= f(2007) + 3 + f(2007) \\ f(2007) &= 3 \end{aligned}$$

■

2) A quadrilateral is drawn in the Cartesian plane with the following co-ordinates: $A = (0,0)$, $B = (16,0)$, $C = (8,8)$, and $D = (0,8)$. A line is drawn connecting A and C. Find the equation of the line parallel to AC which halves the area of the quadrilateral.

Solution: $y = x + 8\sqrt{3} - 16$, rounded $y = x - 2.144$

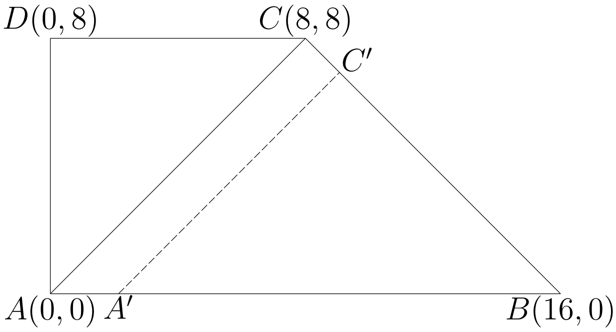
Proof: Draw a picture of the quadrilateral along with the line segment AC.



The area of the quadrilateral $ABCD$ is the sum of the area of the two triangles, so

$$\begin{aligned}\text{area } ABCD &= \text{area } \triangle ACD + \text{area } \triangle ACB \\ &= \frac{1}{2}\cdot 8\cdot 8 + \frac{1}{2}\cdot 16\cdot 8 \\ &= 96\end{aligned}$$

Since area $\triangle ACD <$ area $\triangle ACB$, we have that the line we need to draw to half the area of the triangle is to the right of AC, i.e.



Draw such a line. Let h be the height of$\triangle A'C'B$ and let b be length of the base of $\triangle A'C'B$. Since $AC||A'C'$, we have that $\triangle ACB \sim \triangle A'C'B$. Since in $\triangle ACB$, the base is twice the height, we have that $b = 2h$. That is, the area of triangle $A'C'B$ is $\frac{1}{2}\cdot h\cdot 2h = h^2$. Since we want this line to half the triangle, we have $h^2 = 48$, or $h = 4\sqrt{3}$ while $b = 8\sqrt{3}$. Therefore A' has x intercept $16 - b = 16 - 8\sqrt{3}$, so $A' = (16 - 8\sqrt{3}, 0)$. Since $AC||A'C'$, we have the slope of the line $A'C'$, namely 1. Therefore, we have the slope and a point on the line, so we can find the equation of the line:

$$\begin{aligned}y &= x + b \\ 0 &= 16 - 8\sqrt{3} + b \\ b &= 8\sqrt{3} - 16\end{aligned}$$

Therefore $y = x + 8\sqrt{3} - 16$ or, rounded, $y = x - 2.144$.

■

3) A perfectly cylindrical glass with radius 24 cm and height 50 cm is filled with water. Water is then poured from the glass until the glass is two thirds empty. A solid metal cube is dropped into the glass and the water level rises 3 cm. What is the side length of the cube?

Solution: $(1728\pi)^{\frac{1}{3}} = 12\sqrt[3]{\pi} \sim 17.575$

Proof: Since the glass is $\frac{2}{3}$ empty, it is $\frac{1}{3}$ full. Thus the height of the water is $\frac{50}{3}$, and so the volume of water in the glass is

$$\pi r^2 h = \pi 24^2\left(\frac{50}{3}\right) = 9600\pi$$

After the cube is dropped, the water rises 3 cm, so the new height of water is $\frac{50}{3} + 3$. Thus the new volume of water in the glass is

$$\pi r^2(h+3) = \pi 24^2 \left(\frac{50}{3} + 3\right) = 11328\pi$$

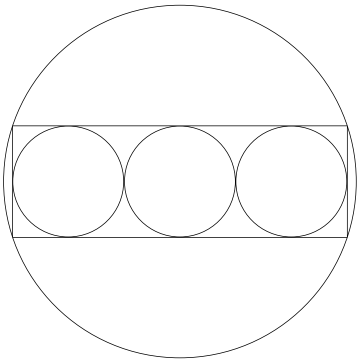
The volume of the cube is the difference between the volume of water with the cube and the volume of water without the cube since the metal cube displaces its volume. That is, the volume of the cube is

$$11328\pi - 9600\pi = 1728\pi$$

Let s be the side length of the cube. Then $s^3 = 1728\pi$, so $s = (1728\pi)^{\frac{1}{3}} = 12\sqrt[3]{\pi} \sim 17.575$

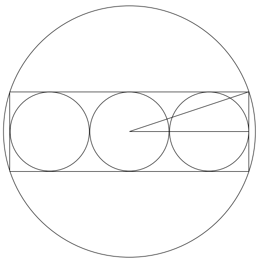
∎

4) Three small circles each have diameter 6. These circles are drawn tangent to each other so their centers form a straight line. They are then inscribed in a rectangle which is inscribed in another larger circle such that the center of the large circle and the center of the small middle circle are the same. What is the area of the large circle (diagram not drawn to scale)?



Solution: $90\pi \sim 282.743$

Proof: Draw a line connecting the center of the large circle to the top right corner of the rectangle and another line from the center of the large circle along the diameter of the small circles to hit the right side of the rectangle.



That is, we have a triangle. The height of the triangle is one half the diameter of the small circle, that is, the height of the triangle is $\frac{6}{2} = 3$. The length of the triangle is the diameter of one small circle plus half the diameter of the middle small circle, i.e. the length is $6 + \frac{6}{2} = 9$. The hypotenuse of the triangle is the radius of the large circle. Call this length r. Then, by Pythagorean formula,

$$r^2 = (3)^2 + (9)^2 = 9 + 81 = 90$$

Then the area of the large circle is $\pi r^2 = \pi 90 \sim 282.743$. ■

5) Alice, Bridget, and Catherine roll a die in that order. Alice wins if she rolls a 1,2, or 3. Bridget wins if she rolls a 4 or 5. Catherine wins if she rolls a six. If no one wins, the die is returned to Alice and the pattern repeated (i.e. Alice, Bridget, Catherine, Alice, etc.). What is the probability that Catherine wins?

Solution: $\frac{1}{13} \sim 0.077$

Proof: Denote the groupings of three as "rotations". The probability that Catherine wins is the probability that she wins on the first rotation plus the probability that she wins on the second rotation plus the probability that she wins on the third rotation plus etc.

Suppose Catherine wins on the first rotation. That is, Alice rolls a 4,5, or 6, Bridget rolls a 1,2,3, or 4, and Catherine rolls a 6. The probability of this happening is $\frac{3}{6} \cdot \frac{4}{6} \cdot \frac{1}{6}$.

Suppose Catherine wins on the second rotation. That is, on the first rotation, Alice, Bridget, and Catherine all lost, then on the second rotation, Alice and Bridget lose while Catherine wins. That is, on the first and second rotations Alice rolls a 4,5, or 6, on the first and second rotations Bridget rolls a 1,2,3, or 4, and on the first rotation Catherine rolls a 1,2,3,4, or 5 while on the second rotation, Catherine rolls a 6. The probability of this happening is $\frac{3}{6} \cdot \frac{4}{6} \cdot \frac{5}{6} \cdot \frac{3}{6} \cdot \frac{4}{6} \cdot \frac{1}{6} = \left(\frac{3}{6}\right)^2 \left(\frac{4}{6}\right)^2 \left(\frac{5}{6}\right) \frac{1}{6}$.

Continuing as such, we see that the probability of Catherine winning on the n^{th} rotation is $\left(\frac{3}{6}\right)^n \left(\frac{4}{6}\right)^n \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$.

Therefore

$$\begin{aligned}
P(\text{Catherine winning}) &= \frac{3}{6} \cdot \frac{4}{6} \cdot \frac{1}{6} + \left(\frac{3}{6}\right)^2 \left(\frac{4}{6}\right)^2 \left(\frac{5}{6}\right) \frac{1}{6} + \left(\frac{3}{6}\right)^3 \left(\frac{4}{6}\right)^3 \left(\frac{5}{6}\right)^2 \frac{1}{6} + \cdots \\
&= \frac{3}{6} \cdot \frac{4}{6} \cdot \frac{1}{6} \left(1 + \frac{3}{6} \cdot \frac{4}{6} \cdot \frac{5}{6} + \left(\frac{3}{6} \cdot \frac{4}{6} \cdot \frac{5}{6}\right)^2 + \cdots\right) \\
&= \frac{1}{18} \left(1 + \frac{5}{18} + \left(\frac{5}{18}\right)^2 + \cdots\right)
\end{aligned}$$

$$= \frac{1}{18}\left(\frac{1}{1-\frac{5}{18}}\right)$$

sum of infinite geometric series

$$= \frac{1}{13}$$

$$\sim 0.077$$

■

6) When expanded, the n^{th} term of $(a + bx + cx^2 + dx^3 + ex^4)(1 - x)^{-5}$ is $n^4 x^{n-1}$. Find $a + b + c + d + e$.

Solution: 24

Proof: We are given that

$$\begin{aligned}
\frac{a + bx + cx^2 + dx^3 + ex^4}{(1-x)^5} &= 1 + 16x + 81x^2 + 256x^3 + 625x^4 + \cdots \\
a + bx + cx^2 + dx^3 + ex^4 &= (1-x)^5(1 + 16x + 81x^2 + 256x^3 + 625x^4 + \cdots) \\
&= (1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5) \\
&\quad\ (1 + 16x + 81x^2 + 256x^3 + 625x^4 + \cdots) \\
&= 1 + (-5x + 16x) + (10x^2 + 81x^2 - 80x^2) \\
&\quad\ + (256x^3 - 405x^3 + 160x^3 - 10x^3) \\
&\quad\ + (625x^4 - 1280x^4 + 810x^4 - 160x^4 + 5x^4) + \cdots \\
&= 1 + 11x + 11x^2 + x^3 + 0x^4 + \cdots
\end{aligned}$$

Therefore, equating coefficients, we obtain $a = 1$, $b = 11$, $c = 11$, $d = 1$, $e = 0$, and so $a + b + c + d + e = 24$. ■