

**Nova Scotia**  

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**Math League**

2007–2008

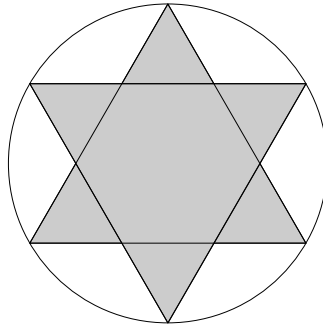
**Provincial Final**

**TEAM QUESTIONS**

## Instructions

- Before the contest begins each team should be assigned a proctor. A team's proctor cannot be their accompanying adult for the sake of impartiality. A good way to do this with minimal fuss is to have each accompanying adult stand next to a team for which they are not the accompanying adult. The adult is then that team's proctor.
- Each question is worth five points with a team scoring either 0 or 5 points on the question (no part marks).
- Five minutes are given to work on each question. At the beginning of the five minutes, the proctor gives each team a copy of the question. At the end of the five minutes, the answer sheet is given to the proctor and the proctor takes it to the front of the room for marking. Students must give up their sheet when time is called. Refusal to do so will result in a mark of zero for the question.
- After handing in their team's response, the proctor then picks up the next question.
- Rounded answers must be correct to three decimal places. Announce this at the beginning.
- If possible, find a team to present their solution. Announce what the correct answer was, find a team with that answer, and invite them to present on an overhead or blackboard.

- 1) The numbers  $1 \leq a, b, c \leq 2008$  are randomly chosen integers (with repetition allowed). What is the probability that  $ab + c$  is even?
- 2) Find the number of subsets of  $\{1, 2, 3, \dots, 62, 63\}$  whose sum of elements is 2008 (note that a one element set  $\{n\}$  has sum  $n$ ).
- 3) Six points are equally spaced along the circumference of a circle of radius 1. Line segments are drawn resulting in the star shape shown below, which is shaded in. Find the area of the shaded region.



- 4) Find the number of times the number 9 appears in the decimal expansion of  $9\,999\,998\,999\,999^2$ .
- 5) Suppose  $a$  is a real number. Find all values of  $a$  such that the quadratic  $x^2 + (a - 3)x + a = 0$  has two distinct, strictly positive (greater than zero), real solutions.
- 6) Six students write a test, sitting in a row from the front of the room to the back of the room. When completed, a student must hand her test into

the proctor at the very front of the room. The length of time it takes any student to finish the test is not affected by the other students in the room.

Suppose

- the students all finish at random times;
- no two students finish at the same time;
- upon finishing, the student stands up immediately to walk to the front;
- the proctor only allows one student to be out of her seat at a time.

What is the probability that some student walking to the front to hand in her test passes by another student who is still writing?

## Answer Key

1)  $\frac{1}{2}$

2) 6

3)  $\sqrt{3} \sim 1.732$

4) 11

5)  $0 < a < 1$

6)  $\frac{719}{720} \sim 0.999$

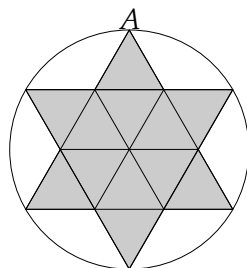
## Solutions

- 1) Note that the probability that  $a$ ,  $b$ , or  $c$  is even is  $\frac{1}{2}$  since half the integers between 1 and 2008 inclusive are even and half are odd.

Consider  $ab + c$ . For  $ab + c$  to be even either  $ab$  and  $c$  are odd or  $ab$  and  $c$  are even. If  $ab$  and  $c$  are odd, then  $a$ ,  $b$ , and  $c$  are odd. The probability of all three being odd is  $\frac{1}{8}$ . Suppose  $ab$  and  $c$  are even. For  $ab$  to be even, either  $a$  is even and  $b$  is odd,  $a$  is odd and  $b$  is even, or  $a$  and  $b$  are even. Each of these occur with probability  $\frac{1}{4}$ . Therefore the probability that  $ab$  is even and  $c$  is even is  $\frac{1}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{8} = \frac{3}{8}$ . Therefore, the probability that  $ab + c$  is even is  $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$ .

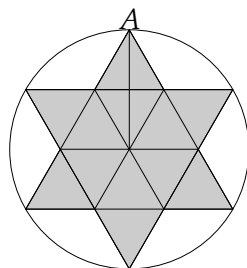
- 2) The sum of all the elements in  $\{1, 2, 3, \dots, 62, 63\}$  is 2016. Let  $S$  be a subset of elements whose sum is 2008. Then the sum of the elements  $\{1, 2, 3, \dots, 63\} \setminus S$  is 8. That is, our question is equivalent to determining the number of subsets of  $\{1, 2, 3, \dots, 62, 63\}$  whose sum of elements is 8. We can find these easily enough by inspection:  $\{8\}$ ,  $\{1, 7\}$ ,  $\{2, 6\}$ ,  $\{3, 5\}$ ,  $\{1, 2, 5\}$ , and  $\{1, 3, 4\}$ .

- 3) Note that the star shape is comprised of twelve identical equilateral triangles:



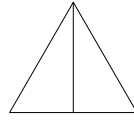
The problem is then reduced to finding the area of one of these triangles.

Consider a line connecting the center of the circle to  $A$ .



This is the radius of the circle and so the line has length 1. It is also the height of two equal sized triangles, so the height of one triangle is 0.5.

Take one of the small triangles:



Let  $x$  be the side length of the triangle. Then the perpendicular height bisects the base, and so, by Pythagorean theorem:

$$x^2 = \left(\frac{1}{2}x\right)^2 + (0.5)^2$$

$$x = \frac{\sqrt{3}}{3}$$

Therefore the area of one of the small triangles is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{12}$$

Since there are twelve triangles each with this area, the area of the shaded region is  $\sqrt{3} \sim 1.732$ .

4) Note that  $9999998999999 = 10^{13} - 10^6 - 1$ . Therefore

$$\begin{aligned} 9999998999999^2 &= (10^{13} - 10^6 - 1)^2 \\ &= 10^{26} - 2 \cdot 10^{19} - 2 \cdot 10^{13} + 10^{12} + 2 \cdot 10^6 + 1 \\ &= 99\ 999\ 979\ 999\ 981\ 000\ 002\ 000\ 001 \end{aligned}$$

There are eleven 9's in the above decimal expansion.

5) Firstly recall that the sum of the roots of the quadratic is  $-(a - 3)$  (the co-efficient of the  $x$  term) while the product of the roots of the quadratic is  $a$  (the co-efficient of the constant term). If  $a < 0$ , then the product of the roots of the quadratic is negative, so one of the roots would be negative. Therefore, we need  $a > 0$ . Suppose  $-(a - 3) < 0$ . That is, the sum of the roots would be negative, which would mean one of the roots would be negative. Therefore  $-(a - 3) > 0$ . Rearranging we obtain  $a < 3$ . That is, if  $a$  is not strictly between 0 and 3, then there is one negative root. Therefore,  $0 < a < 3$ .

Now we consider having two distinct, real roots. This occurs when the discriminant in the quadratic formula is strictly positive. That is, when  $(a - 3)^2 - 4a > 0$ . Expanding, this is  $(a - 9)(a - 1) > 0$ . Now  $(a - 9)(a - 1)$  is an upwards facing parabola which is strictly positive when  $a < 1$  or  $a > 9$ .

Therefore, we are looking for  $a$ 's which are both  $0 < a < 3$  and  $(a < 1$  or  $a > 9)$ . The only such  $a$  which satisfy both these conditions are  $0 < a < 1$ .

6) Consider the number of different orderings of the students to finish their tests. There are  $6! = 720$  different orderings. Of these, only one ensures that no student passes by another student, namely that the student at the first desk finishes first, the student at the second desk finishes second, . . . , the student at the sixth desk finishes sixth. Thus, the probability that no student passes by another student still writing is  $\frac{1}{720}$ . Therefore the probability that at least one student passes by another student still writing is  $1 - \frac{1}{720} = \frac{719}{720}$ .