Nova Scotia

Math League

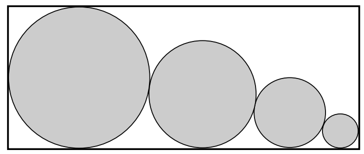
2007–2008

Game Two

Team Questions

Problems

1) Four circles fit perfectly into a rectangle as shown in the figure. If the circles have radii 1cm, 2cm, 3cm and 4cm, what is the area of the rectangle?



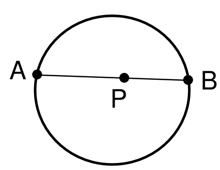
2) An integer is chosen at random between 1 and 999, inclusive. Find the probability that it does not contain the digit 5.

3) Find the area of the triangular region of the plane bounded by the lines $2y - x = 0$, $y - 3x = 0$ and $y + x = 4$.

4) The campers at a summer camp are 13 years old and the counselors are 17 years old. If the total age of all campers and counselors is 505, what is the maximum possible number of campers attending the camp?

5) The equation $x^2 + x - 2008 = 0$ has two nonzero real roots, a and b. Give a quadratic equation with integer coefficients whose roots are $\frac{1}{a}$ and $\frac{1}{b}$.

6) A circular lake of diameter 100 m. Alan and Bob stand on the shore of the lake at points A and B, respectively. The boys jump in the water simultaneously and begin swimming directly towards one another, with Alan moving at 1.25 m/sec and Bob moving at 1 m/sec. The boys meet after exactly 40 seconds at the point P. How far is P from the centre of the lake?



7) A *positive divisor* of an integer n is a positive integer d that divides evenly into n. For example, the positive divisors of 28 are 1, 2, 4, 7, 14, and 28.

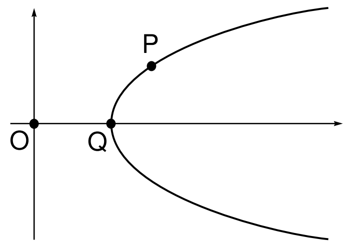
   How many integers between 1 and 1000 inclusive have exactly 5 positive divisors?

8) Find all pairs (x, y) of positive integers such that

$$x^2 + y - 101 = y^2 - x + 101.$$

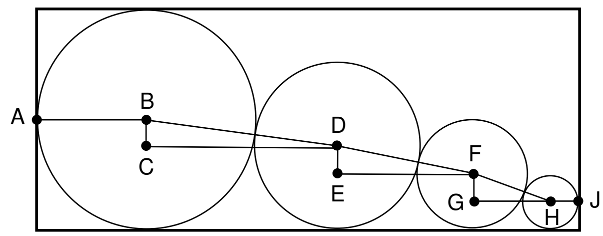
9) Two points are chosen at random on the perimeter of a circle of radius 1 cm. What is the probability that the chord joining these two points has length at least 1 cm?

10) Let $O = (0, 0)$ and $Q = (4, 0)$. Find the coordinates of the point P on the parabola $x = y^2 + 4$ for which angle $\angle POQ$ is greatest.



Solutions

1) Clearly the height of the rectangle is the diameter of the largest circle, which is 8 cm. Consider the diagram below.

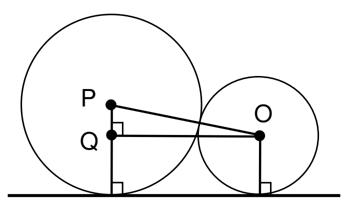


Clearly $|AB| = 4$ and $|HJ| = 1$, so the width of the rectangle is $5 + |CD| + |EF| + |GH|$. But $|BC| = |DE| = |FG| = 1$, and $|HF| = 1 + 2 = 3$, $|FD| = 2 + 3 = 5$, $|DB| = 3 + 4 = 7$. So Pythagorean theorem gives

$$|CD| = \sqrt{7^2 - 1^2} = 4\sqrt{3}$$
$$|EF| = \sqrt{5^2 - 1^2} = 2\sqrt{6}$$
$$|GH| = \sqrt{3^2 - 1^2} = 2\sqrt{2}.$$

Hence the area of the rectangle is $8(5 + 2\sqrt{2} + 4\sqrt{3} + 2\sqrt{6})$ cm^2.

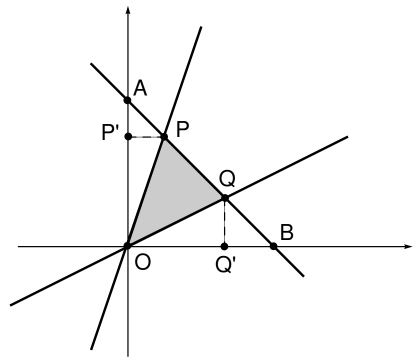
Note: Whenever two circles of radius r and $r + 1$ are touching as shown below, we have $|PQ| = 1$ and $|OP| = r + (r + 1) = 2r + 1$, whence $|OQ| = \sqrt{(2r+1)^2 - 1^2} = \sqrt{4r^2 + 4} = 2\sqrt{r(r+1)}$.



To apply this in the situation above we set $r = 1, 2, 3$ to quickly get $|GH| = 2\sqrt{1 \cdot 2}$, $|EF| = 2\sqrt{2 \cdot 3}$, and $|CD| = 2\sqrt{3 \cdot 4}$. If the problem had involved more circles, this general formula would be much handier than repeating the Pythagorean calculation many times.

2) A number between 1 and 999 that does not contain the digit 5 is of the form ABC, where each of A, B, and C is any of $\{0, 1, 2, 3, 4, 6, 7, 8, 9\}$, but not all of A,B,C are 0. Thus there are $9 \cdot 9 \cdot 9 - 1 = 728$ such numbers, and the required probability is $\frac{728}{999}$.

3) See the diagram below. Clearly $A = (4, 0)$, $B = (0, 4)$, and lines $\{y - 3x = 0, y + x = 4\}$ meet at $P = (1, 3)$ while $\{2y - x = 0, y + x = 4\}$ meet at $Q = (\frac{8}{3}, \frac{4}{3})$.



We therefore find:

$$\begin{aligned}
\text{area}(\triangle OPQ) &= \text{area}(\triangle OAB) - \text{area}(\triangle OAP) - \text{area}(\triangle OQB) \\
&= \tfrac{1}{2}|OA||OB| - \tfrac{1}{2}|OA||PP'| - \tfrac{1}{2}|OB||QQ'| \\
&= \tfrac{1}{2} \cdot 4 \cdot 4 - \tfrac{1}{2} \cdot 4 \cdot 1 - \tfrac{1}{2} \cdot 4 \cdot \tfrac{4}{3} \\
&= \tfrac{10}{3}.
\end{aligned}$$

Alternative solution: The area of the triangle with vertices $O = (0, 0)$, $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ is given by the determinant

$$\begin{aligned}
\text{area}(\triangle OPQ) &= \frac{1}{2}\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \\
&= \tfrac{1}{2}(p_1 q_2 - p_2 q_1).
\end{aligned}$$

This useful fact can be proved simply by generalizing the work above. Check that setting $P = (1, 3)$ and $Q = (\frac{8}{3}, \frac{4}{3})$ yields an area of $\frac{10}{3}$, in agreement with our other calculation.

Note: Suppose one wishes to find the area of a triangle with given vertices P, Q and R, where none of these is the origin. Then one simply shifts the triangle so one of the vertices, say R, *is* the origin. Of course, this changes the coordinates of P and Q, but the shift does not affect the area of the triangle, and the formula above can be applied.

4) Let x and y be the number of campers and counselors, respectively. Then $13x + 17y = 505$, and we wish to determine the maximal integer value of x such that this equation holds for some integer y.

Notice that $13x + 17y = 505$ implies $x = \frac{1}{13}(505 - 17y)$. For x to be integral and as large as possible, we want y as small as possible such that $11 - 4y$ is divisible by 13. Checking

$y = 1, 2, 3, \ldots$, we quickly find that $y = 6$ is the first such value. Therefore there are at most $x = \frac{1}{13}(505 - 17 \cdot 6) = 31$ campers and at least $y = 6$ counselors.

Note: This solution is somewhat unsatisfactory, since it relies on trial and error with potentially tedious divisions. Perhaps we got lucky: what if the first good value of y wasn't as small as $y = 6$?

It turns out we were a little lucky, but not extraordinarily so. A little work with *modular arithmetic* shows that $y = 12$ would be the worst case scenario. It also allows the tedious divisions to be replaced by much simpler modular computations.

Nonetheless, the method in the following alternative solution avoids guessing altogether. In fact, it can be applied to find all integer solutions (x, y) of $ax + by = c$, where a, b, c are any given integers. (These are called *linear Diophantine equations*.)

Alternative Solution: As before, write $x = \frac{1}{13}(505 - 17y)$, but divide with remainder to write this as

$$x = 38 - y + \frac{11 - 4y}{13}.$$

For x to be integral, $11 - 4y$ must be divisible by 13. That is, $11 - 4y = 13w$ for some integer w, and this leads to

$$y = \frac{11 - 13w}{4} = 2 - 3w + \frac{3 - w}{4}.$$

Since y is integral, $3 - w$ is divisible by 4, whence $3 - w = 4z$ for some integer z. Thus $w = 3 - 4z$, and reverse substitution yields $y = 13z - 7$ then $x = 48 - 17z$. So all integer solutions of $13x + 17y = 505$ are of the form $(x, y) = (48 - 17z, -7 + 13z)$ for some integer z. We want x as big as possible while y is positive. Clearly this occurs when $z = 1$ and $(x, y) = (31, 6)$.

5) If $x \neq 0$, then division by x^2 gives:

$$x^2 + x - 2008 = 0 \quad \Longrightarrow \quad 1 + \frac{1}{x} - \frac{2008}{x^2} = 0 \quad \Longrightarrow \quad 2008\left(\frac{1}{x}\right)^2 - \frac{1}{x} - 1 = 0.$$

So if a, b are roots of $x^2 + x - 2008 = 0$, then $\frac{1}{a}$ and $\frac{1}{b}$ are roots of $2008x^2 - x - 1 = 0$.

Alternative Solution: We have $x^2 + x - 2008 = (x - a)(x - b) = x^2 - (a + b)x + ab$. Hence $ab = -2008$ and $a + b = -1$. Note that $(x - \frac{1}{a})(x - \frac{1}{b}) = 0$ has roots $\frac{1}{a}$ and $\frac{1}{b}$. But

$$\begin{aligned}(x - \tfrac{1}{a})(x - \tfrac{1}{b}) &= x^2 - (\tfrac{1}{a} + \tfrac{1}{b})x + \tfrac{1}{ab} \\ &= x^2 - \frac{a+b}{ab}x + \frac{1}{ab} \\ &= x^2 - \left(\frac{-1}{-2008}\right)x + \left(\frac{1}{-2008}\right) \\ &= x^2 - \frac{1}{2008}x - \frac{1}{2008}.\end{aligned}$$

We multiply by 2008 to force this equation to have integer coefficients without changing its roots. This yields the equation $2008x^2 - x - 1 = 0$.

Note: The method used in the alternative solution can be applied in many more general situations. For instance, to find a quadratic equation with roots a^2 and b^2, we note that

$$(x - a^2)(x - b^2) = x^2 - (a^2 + b^2) + (ab)^2$$

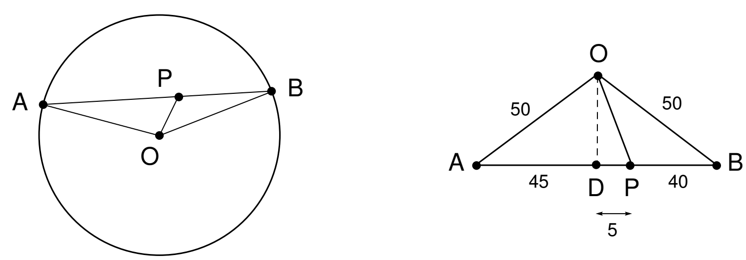
is such an equation, and then use the fact that $(ab)^2 = (-2008)^2$ and

$$a^2 + b^2 = (a + b)^2 - 2ab = (-1)^2 - 2(-2008) = 4017.$$

Hence $x^2 - 4017x + 2008^2 = 0$ has roots a^2 and b^2.

How could this method could be used to find a quadratic whose roots are a^3 and b^3? How could one find a cubic polynomial whose roots are α^2, β^2 and γ^2, where α, β, γ are the roots of $x^3 - 3x + 1 = 0$?

6) The situation is illustrated below. In 40 seconds, Alan swims $|AP| = 40 \cdot 1.25 = 50$ metres while Bob swims $|BP| = 40 \cdot 1 = 40$ metres. Letting O be the centre of the circle, we have $|OA| = |OB| = 50$. We wish to determine $|OP|$.



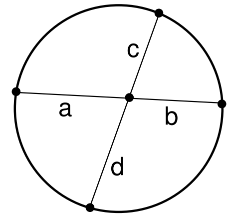
Since $\triangle OAB$ is isosceles, a perpendicular from O meets AB at its midpoint D. That is, $|AD| = |DB| = 45$, whence $|DP| = 5$. Then right triangle $\triangle OAD$ gives $|OD|^2 + 45^2 = 50^2$, while right triangle $\triangle ODP$ yields $|OD|^2 + 5^2 = |OP|^2$. Hence $|OP|^2 = 5^2 + 50^2 - 45^2 = 500$, so $|OP| = 10\sqrt{5}$.

Alternative Solution: Let P be any point inside a circle of radius r, and let d be the distance from P to the centre of the circle. Then the quantity $r^2 - d^2$ is called the *power of the point* P, and a very nice fundamental result of circle geometry is the following: For any chord passing through P and cutting the circle at points A and B, we have

$$r^2 - d^2 = |AP| \cdot |PB|. \qquad (*)$$

In our problem, $r = 50$, $|AP| = 50$ and $|PB| = 40$, and we wish to determine d. The power of the point formula gives $50^2 - d^2 = 50 \cdot 40$. Hence $d = \sqrt{50^2 - 40 \cdot 50} = 10\sqrt{5}$.

Note: The amazing thing about equation ($*$) is that it holds for *any* chord passing through P. Since r and d are constant, this implies that $|AP| \cdot |PB|$ is the same for any chord passing through P (where A and B are the points at which the chord meets the circle). In particular, in the diagram below we always have $ab = cd$.



The power of the point formula can be proved by generalizing the argument used in our first solution, but it is better seen as a *consequence* of the fact that $ab = cd$ in the diagram above. This is called the *intersecting chords theorem*. It has a very simple proof based on similar triangles and some circle geometry. Ask your teacher!

7) Let n be a positive integer with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. (That is, the p_i are distinct prime numbers and each a_i is positive.) Then every positive divisor of n is of the form $p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ where $0 \le b_i \le a_i$. Since there are $a_i + 1$ possibilities for each exponent b_i, it follows that n has exactly $(a_1+1)(a_2+1)\cdots(a_k+1)$ positive divisors. For example, $28 = 2^2 \cdot 7^1$ has $(2+1)(1+1) = 6$ positive divisors.

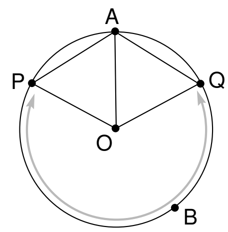
   For exactly 5 prime divisors, we would require $(a_1+1)(a_2+1)\cdots(a_k+1) = 5$. But 5 itself is prime and the a_i are positive, so this forces $k = 1$ and $a_1 = 4$. That is, $n = p^4$ for some prime p. Thus the given problem reduces to finding the number of perfect fourth prime powers between 1 and 1000. There are only 3 of these, namely $2^4, 3^4$, and 5^4.

8) Rearrange and factor to get

$$x^2 - y^2 + x + y = 202 \iff (x-y)(x+y) + y + x = 202$$
$$\iff (x+y)(x-y+1) = 2 \cdot 101.$$

   Since x and y are positive integers, $x - y + 1 < x + y$. Since 2 and 101 are prime this forces $\{x+y = 202, x-y+1 = 1\}$ or $\{x+y = 101, x-y+1 = 2\}$. These systems yield $(x,y) = (101,101)$ or $(x,y) = (51,50)$.

9) Let the first chosen point be A, and let P and Q be the points on the circle exactly 1 cm from A. The second chosen point, B, will be at least 1 cm from A provided it lies on the circle between P and Q, as indicated in the diagram below



Let O be the centre of the circle. Then $\triangle OPA$ and $\triangle OQA$ are equilateral, so $\angle POA = \angle QOA = 60^\circ$ and arc PAQ is subtended by a central angle of 120°. This leaves a 240° arc along which B can be chosen (highlighted with a shaded arrow above). The desired probability is therefore $\frac{240}{360} = \frac{2}{3}$.

10) Clearly $\angle POQ$ is maximal when line OP meets the parabola exactly once; that is, when OP is tangent to the parabola. If OP has slope α, then it has equation $y = \alpha x$. Substitute into $x = y^2 + 4$ to get the quadratic $\alpha^2 x^2 - x + 4 = 0$. This has only one root when its discriminant $1 - 16\alpha^2$ is 0, which occurs when $\alpha = \pm\frac{1}{4}$. In this case, we get $x = \frac{1}{2\alpha^2} = 8$. Then $y = \alpha x = \pm\frac{1}{4} \cdot 8 = \pm 2$. Thus $P = (8, 2)$.

Alternative Solution 1: This solution uses calculus. As above, $\angle POQ$ is maximal when OP is tangent to the parabola. Since $x = y^2 + 4$, we have $1 = 2y \cdot \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{2y}$. Thus the equation of the tangent to the parabola at the point $(a^2 + 4, a)$ is

$$y - a = \frac{1}{2a}(x - a^2 - 4).$$

We want this line to pass through the origin, so we set $x = y = 0$ to get $-a = \frac{1}{2a}(-a^2 - 4)$. This yields $a = 2$, and therefore $P = (2^2 + 4, 2) = (8, 2)$.

Alternative Solution 3: Clearly $\angle POQ$ is greatest when the slope of OP is maximal. Since P lies on the parabola, it must have coordinates $(y^2 + 4, y)$ for some y. Then the slope of segment OP is

$$\frac{y}{y^2 + 4}.$$

and to maximize this quantity we minimize the reciprocal

$$\frac{y^2 + 4}{y} = y + \frac{4}{y}.$$

Since y is clearly positive, the arithmetic-geometric mean inequality gives

$$\frac{y + \frac{4}{y}}{2} \leq \sqrt{y \cdot \frac{4}{y}} \quad \Longrightarrow \quad y + \frac{4}{y} \leq 4.$$

Equality holds precisely when $y = \frac{4}{y}$, that is, when $y = 2$. Setting $y = 2$ in $x = y^2 + 4$ yields $x = 8$, so that $P = (8, 2)$.

Alternative Solution 4: Follow Alternative Solution 3, but use calculus to either maximize the quantity $y/(y^2+4)$ or minimize $y + \frac{4}{y}$.

Note: The *arithmetic-geometric mean inequality* (AGM) is one of the most important inequalities of mathematics. In its most basic form it states that for positive real numbers a, b, we have

$$\frac{a+b}{2} \le \sqrt{ab},$$

with equality holding if and only if $a = b$. This is easily proved by noting that

$$a - 2\sqrt{ab} + b = (\sqrt{a} - \sqrt{b})^2 \ge 0.$$

The right-hand inequality is clearly strict except in the case $\sqrt{a} = \sqrt{b}$.

The generalization of the AGM to many variables states that, for positive real numbers $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \le \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.