Nova Scotia Math League

2007–2008

Game Three

TEAM QUESTIONS

Problems

1) Given $x, y \geq 0$, $xy = 48$ and $x^2 + y^2 = 1026.25$, find $x + y$.

2) A drawer contains 37 socks. Of these, 26 are red and 11 are green. Two socks are called a *matched pair* if they are the same colour. Blindfolded, you reach in and pull out five socks. What is the probability that you have two matched pairs of socks?

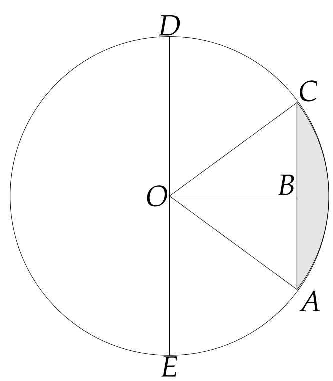
3) Function Machine `A` takes in a number, squares it, then adds one, and outputs the result. Function Machine `B` takes in a number, multiplies it by 2, then adds 5, and outputs the result.

   A number is input into Machine `A`. The output from Machine `A` is then input into Machine `B`. The output from Machine `B` is 119.5. What number was initially put into Machine `A` to obtain the final result?

4) The circle below has centre O and diameter 4. DE is a diameter of the circle. A, B, and C are chosen so that

$$\angle AOE = \angle AOB = \angle BOC = \angle COD.$$

   Find the area of the shaded region (diagram not drawn to scale).



5) At the NSML event in Halifax, the ratio of males to females was 3 : 4. At the NSML event in Sydney, the ratio of males to females was 4 : 5. Both regions had the same number of students participate.

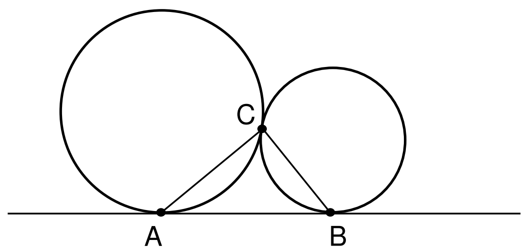
   Find, in lowest terms, the ratio of males who participated in the Halifax event to the males who participated in the Sydney event.

6) Three hunters shoot at a target. Each of the hunters takes one shot, and each has a one in three chance of hitting the target. What is the probability that at least one of the hunters hits the target?

7) A *perfect square* is a number of the form n^2, where n is a positive integer. How many numbers in the following infinite set are perfect squares?

$$\{1,2,3,4,5,6,7,8,9,$$
$$11,22,33,44,55,66,77,88,99,$$
$$111,222,333,444,555,666,777,888,999,\ldots\}$$

8) Two circles, with radii 9 cm and 16 cm, are tangent at a point C. A common tangent meets these circles at A and B, as in the diagram below. Find the area of triangle $\triangle ABC$.



9) Two real numbers b and h are chosen randomly between 0 and 1, inclusive, and a right triangle is constructed with base b and height h. What is the probability that the hypotenuse of this triangle has length less than 1?

10) Consider the equation

$$2^{12}+4^6+8^4+16^3=a^b.$$

For how many pairs (a,b) of positive integers does this hold true?

Solutions

1) Notice that $(x+y)^2 = x^2 + 2xy + y^2$. Therefore

$$(x+y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2(xy) = 1026.25 + 2(48) = 1122.25$$

Taking the square root gives us that $x + y = \pm 33.5$. Since both $x, y \geq 0$, this means $x + y = 33.5$.

2) With R representing red socks and G representing green socks, the possibilities for sock colour are as follows:

$$(5R, 0G), (4R, G), (3R, 2G), (2R, 3G), (R, 4G), (0R, 5G).$$

In each of these possibilities, there are two matched pairs of socks. Since one of these cases must occur and each case has the desired result (two matched pairs), the probability of drawing two matched pairs is 100%.

3) Let x be the number initially put into Machine `A`. Then the output from Machine `A` is $x^2 + 1$. Call this number y. Then y is then put into Machine `B`. The output from Machine `B` is $2y + 5$. Therefore

$$\begin{aligned} 119.5 &= 2y + 5 \\ 119.5 &= 2(x^2 + 1) + 5 \\ 114.5 &= 2(x^2 + 1) \\ 57.25 &= x^2 + 1 \\ 56.25 &= x^2 \\ x &= \pm 7.5. \end{aligned}$$

4) The radius of the circle is 2. The area of the semi-circle is 2π.

Since DE is a straight line, $\angle DE = 180°$. Since A, B, and C are chosen so that

$$\angle AOE = \angle AOB = \angle BOC = \angle COD$$

and we have that

$$180° = \angle DE = \angle AOE + \angle AOB + \angle BOC + \angle COD,$$

we combine these two facts to get

$$45° = \angle AOE = \angle AOB = \angle BOC = \angle COD.$$

Thus the section of the circle AOE is one eighth of the area of the total circle, one fourth the area of the semi circle. That it, section AOE has area $\frac{\pi}{2}$. Similarly, section COD has area $\frac{\pi}{2}$.

Consider $\triangle AOC$. Since $\angle AOC = \angle AOB + \angle BOC$, we have that $\angle AOC = 90°$. Since AO and OC are radii of the circle, $|AO| = |OC| = 2$. That is, we have that $\triangle AOC$ is a right angle triangle with non-hypotenuse sides both having length 2. Therefore, the area of $\triangle AOC$ is 2.

Let X be the area of the shaded region. Then the area of the semi-circle is the sum of X, the area of $\triangle AOC$, the area of section COD, and the area of section AOE. That is:

$$2\pi = X + \frac{\pi}{2} + 2 + \frac{\pi}{2}$$
$$X = \pi - 2$$

5) Let X be the total number of students at either event. Since the ratio of males to females at the Halifax event was $3 : 4$, then there were $3h + 4h$ students who participated in the Halifax event, for some positive integer h. Since the ratio of males to females at the Sydney event was $4 : 5$, then there were $4s + 5s$ students who participated at the Sydney event, for some positive integer s. Then

$$3h + 4h = 7h = X = 9s = 4s + 5s.$$

Therefore,

$$7h = 9s$$
$$h = \frac{9}{7}s.$$

The ratio of males in Halifax to males in Sydney is

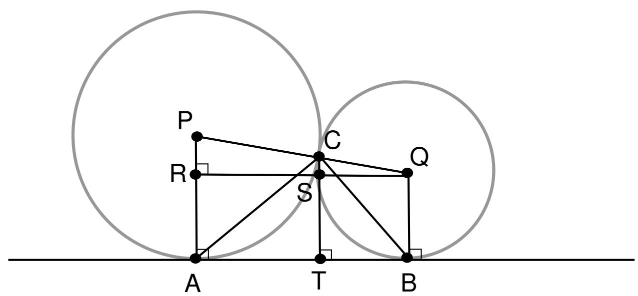
$$3h : 4s$$
$$3\left(\frac{9}{7}s\right) : 4s$$
$$27s : 28s$$
$$27 : 28$$

6) The desired probability is 1 less the probability that no hunter hits the target. Since each hunter shoots independently of the others, and each has a $\frac{2}{3}$ chance of missing, the probability that no hunter hits the target is $(\frac{2}{3})^3 = \frac{8}{27}$. This gives an answer of $1 - \frac{8}{27} = \frac{19}{27}$.

7) The numbers 1, 4, and 9 are the only three perfect squares in the given set, giving an answer of 3.

To see this, first observe that perfect squares always terminate in the digits 0, 1, 4, 5, 6, or 9. Thus all numbers involving other digits are excluded from consideration. Then note that the number $n = XXX \cdots X$ (where X represents a digit $1, 4, 5, 6$, or 9) can be factored as $n = X \cdot (111 \cdots 1)$. If $X = 5$ or $X = 6$, then n cannot be a square, since clearly $111 \cdots 1$ is not divisible by 5 or 6. Moreover, the only number of the form $111 \cdots 1$ that is a perfect square is 1 itself, since all other such numbers leave a remainder of 3 upon division by 4. Since $X = 1, 4, 9$ are perfect squares, it follows that $n = X \cdot (111 \cdots 1)$ is *not* a perfect square unless $n = 1, 4, 9$.

8) Consider the set-up shown in the diagram below. Since $|QB| = 9$ and $|PA| = 16$, we have $|ST| = |QB| = 9$, $|PR| = 16 - 9 = 7$ and $|PQ| = 9 + 16 = 25$.



From right triangle $\triangle PQR$, Pythagorean theorem gives

$$|RQ|^2 + |PR|^2 = |PQ|^2 \Longrightarrow |RQ|^2 = 25^2 - 7^2 \Longrightarrow |RQ| = 24.$$

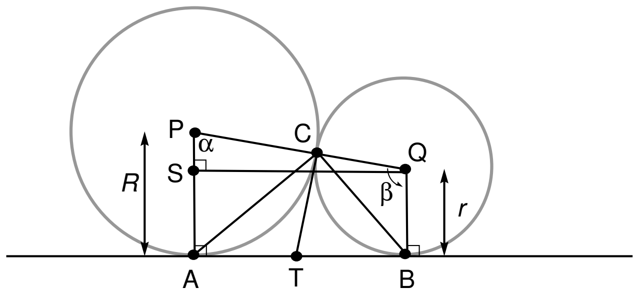
Hence $|AB| = 24$. Moreover, $\triangle CQS$ is similar to $\triangle PQR$, so we have

$$\frac{|CS|}{|CQ|} = \frac{|PR|}{|PQ|} \Longrightarrow \frac{|CS|}{9} = \frac{7}{25} \Longrightarrow |CS| = \frac{63}{25}.$$

Thus $|CT| = |CS| + |ST| = \frac{63}{25} + 9 = \frac{288}{25}$. The area of $\triangle ABC$ is then

$$\frac{1}{2}|AB| \cdot |CT| = \frac{1}{2} \cdot 24 \cdot \frac{63}{25} = \frac{3456}{25}.$$

Alternative Solution: Let $\alpha = \angle APC$ and $\beta = \angle CQB$, and for simplicity let the radii of the circles be R and r, as in the diagram below.



Suppose the common tangent two the circles at C meets AB at T. Then $\triangle TAC$ and $\triangle TCB$ are isosceles, whence $|TA| = |TB| = |TC|$. That is, C lies on the circle with centre T and diameter $|AB|$. It follows that $\triangle ABC$ is right angled (with $\angle ACB = 90°$), so its area is $\frac{1}{2}|AC| \cdot |BC|$.

Cosine law applied to $\triangle PAC$ and $\triangle QBC$ gives

$$|AC|^2 = R^2 + R^2 - 2R^2 \cos\alpha = 2R^2(1 - \cos\alpha)$$
$$|BC|^2 = r^2 + r^2 - 2r^2 \cos\beta = 2r^2(1 - \cos\beta).$$

But AP is parallel to BQ, so $\beta = 180° - \alpha$ and thus $\cos\beta = -\cos\alpha$. Multiplying the two identities above and taking a square root then yields

$$|AC| \cdot |BC| = 2Rr\sqrt{1 - \cos^2\alpha}.$$

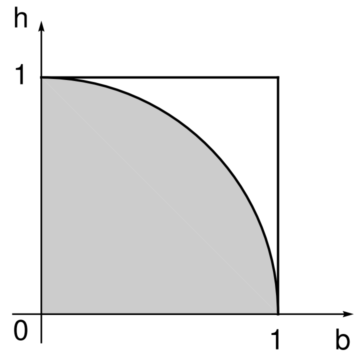
But from right triangle $\triangle PSQ$ we have $\cos\alpha = |PS|/|PQ| = \frac{R-r}{R+r}$. Hence

$$\text{area}(\triangle ABC) = \frac{1}{2}|AC| \cdot |BC| = Rr\sqrt{1 - \left(\frac{R-r}{R+r}\right)^2} = \frac{2Rr\sqrt{Rr}}{R+r}.$$

Setting $R = 16$, $r = 9$ again shows the area is $\frac{3456}{25}$.

9) The possible pairs (b, h) comprise the unit square $\{(b, h) : 0 \leq b, h \leq 1\}$ when plotted in the (b, h)-plane. The hypotenuse of the right-triangle with base b and height h has length $\sqrt{b^2 + h^2}$, and this is less than 1 precisely when $b^2 + h^2 < 1$.

That is, the hypotenuse is of length less than 1 when the point (b, h) lies in shaded region shown below.



We want to find the probability that a randomly chosen point (b, h) in the square lies within this shaded region. But the shaded region is just a quarter of a circle with radius 1, so its area is $\frac{\pi}{4}$. Since the area of the unit square is 1, the desired probability is $\frac{\pi/4}{1} = \frac{\pi}{4}$.

10) Since $2^{12} = 4^6 = 8^4 = 16^3$, we have $2^{12} + 4^6 + 8^4 + 16^3 = 4 \cdot 2^{12} = 2^{14}$. The resulting equation $a^b = 2^{14}$ has exactly 4 solutions, namely $(a, b) = (2^1, 14), (2^2, 7), (2^7, 2)$, and $(2^{14}, 1)$.