Nova Scotia

Math League

2007–2008

Game One

PAIRS RELAY

Problems

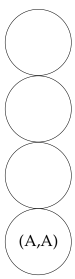
A. The polynomial $x^2 + 275x - 127$ has two real roots. Let A be the product of their roots.

Pass on A.

B. You will receive A.

A circle of radius 3 is centered at the point (A,A) in the (x, y)-plane. Circles of radius 3 are stacked on top of this circle, each one tangent to the circles immediately above and below such that the line connecting their centers is parallel to the y-axis. (See diagram.)

The circles are numbered by letting the original circle centered at (A,A) be Circle 1, and counting upward from there. Let Circle B be the first circle which intersects the x-axis.



Pass on B.

C. You will receive B.

A square of side length B is divided into 9 smaller congruent squares by equally placed lines parallel to the sides. A circle is inscribed in each small square. Let $C\pi$ be the sum of the areas of all of the inscribed circles.

Pass on C.

D. You will receive C.

Four roads form a perfect square with perimeter 7744 meters. A tortoise starts at one corner and walks around the square without stopping until he returns to his starting position. On the first side, he walks at C metres/hour. On each successive side, he goes twice as fast as on the previous side. Let D be the average speed of his entire trip **rounded to three decimal places**.

Pass on D. (Remember to round to three decimal places!)

Solutions

A. If a and b are the roots, then

$$x^2 + 275x - 127 = (x-a)(x-b) = x^2 - (a+b)x + ab$$

Equating the two sides, $ab = -127$.

B. Each circle has radius 3, so diameter 6. We are starting at A, adding 3 for the radius of Circle 1, and then 6 for each successive circle. We want to know when this sum is greater than or equal to zero, corresponding to when a circle intersects the x-axis. That is, we want the smallest integer n such that

$$A + 3 + 6n \geq 0,$$

where n corresponds to the number of circles above the original circle. This inequality is equivalent to $n \geq -\frac{1}{6}(A+3)$, and setting $A = -127$ gives $n \geq 20\frac{2}{3}$. Hence $n = 21$. It follows that $B = 21 + 1 = 22$, where the $+1$ accounts for the initial circle.

C. Each square has side length $\frac{B}{3}$. Therefore each circle has radius $\frac{B}{6}$. Therefore each circle has area $\frac{B^2}{36}\pi$. Since there are 9 circles, the sum of the areas of all the circles is $\frac{B^2}{4}\pi$.

Since $B = 22$, the sum of the areas of the circles is 121.

D. Average speed is total distance divided by total time. Note that the length of each side of the square is irrelevant to this calculation, since the square could be scaled by any factor and this quantity would remain the same. So we simplify our calculations by assuming the square has side length 1 unit.

On the first side, the tortoise walks C units per hour, so it takes him $\frac{1}{C}$ hours to walk that side. On the next side, the tortoise walks 2C units/hour, so it takes him $\frac{1}{2C}$ hours to walk that side. Similarly it takes the tortoise $\frac{1}{4C}$ and $\frac{1}{8C}$ hours to talk the third and fourth sides, respectively.

Therefore his total travel time is

$$\frac{1}{C} + \frac{1}{2C} + \frac{1}{4C} + \frac{1}{8C} = \frac{1}{C}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8C}.$$

Since the total distance is 4 units, the average speed is

$$D = \frac{4}{\frac{15}{8C}} = \frac{32C}{15}.$$

Since $C = 121$, we have $D = \frac{32 \cdot 121}{15} = \frac{3872}{15} \sim 258.133$.