Nova Scotia

Math League

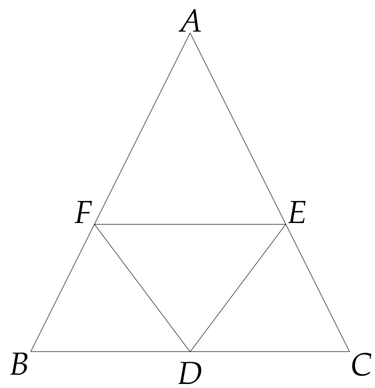
2007–2008

Game One

TEAM QUESTIONS

Problems

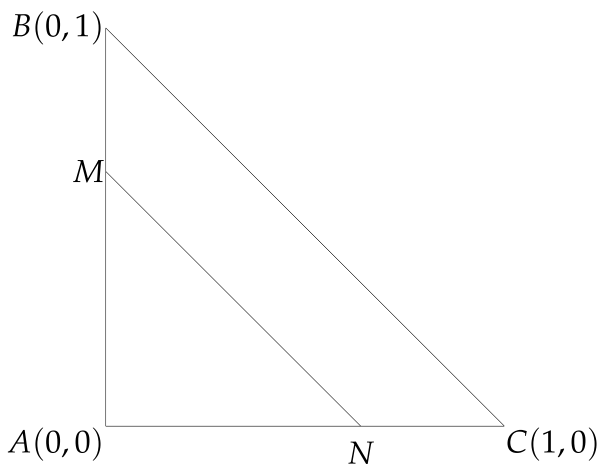
1) $\triangle ABC$ is an isosceles triangle with $\angle ABC = \angle ACB$. Points D, E, and F are chosen on BC, CA, and AB respectively such that $\triangle DEF$ is equilateral. If $\angle AFE = 82^\circ$ and $\angle CED = 86^\circ$, find angle $\angle BDF$. (Diagram not drawn to scale.)



2) $\triangle ABC$ lies in the plane with $A = (0,0)$, $B = (0,1)$ and $C = (1,0)$. Points M and N are chosen on AB and AC, respectively, such that

- MN is parallel to BC.
- MN divides the area of $\triangle ABC$ in half.

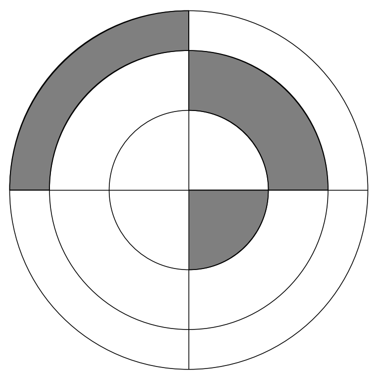
Find the co-ordinates of M. (Diagram not drawn to scale.)



3) At Meghan's family reunion, one third of the female guests did not speak French. However, half the male guests spoke French. If there were three-quarters as many women as men who attended, find the ratio in lowest terms of guests who spoke French to the total number of guests.

4) A new subdivision is being built and Harry is in charge of putting numbers on the doors. The homes are to be numbered consecutively, beginning with 1, using brass numerals. For example, house #22 uses two brass numerals, while #149 uses three numerals. If Harry determines that 999 such numerals are needed to complete his task, how many houses are in the subdivision?

5) In the diagram (not drawn to scale), two perpendicular lines intersect at the center of three concentric circles. Each shaded region has the same area. If the radius of the smallest circle is 1, find the product of the radii of the three circles.

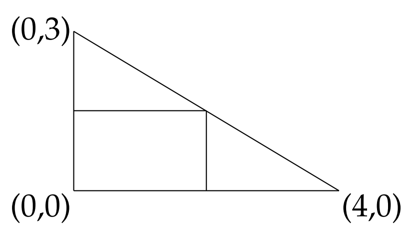


6) A function f goes from the positive integers (integers strictly greater than zero) to the positive integers such that

$$f(n) = \begin{cases} \frac{n}{3} & \text{if } n \text{ divisible by 3} \\ 2n+1 & \text{if } n \text{ is not divisible by 3} \end{cases}$$

For example, $f(4) = 9$, while $f(9) = 3$. For how many positive integers is it true that $f(f(n)) = n$?

7) What is the area of largest rectangle that can be inscribed in a right angle triangle with non-hypotenuse side lengths 3 and 4? (The diagram is not drawn to scale.)



8) Alice has nine coins and Bridget has eight. All coins are fair and are tossed simultaneously. What is the probability that Alice tosses more heads than Bridget?

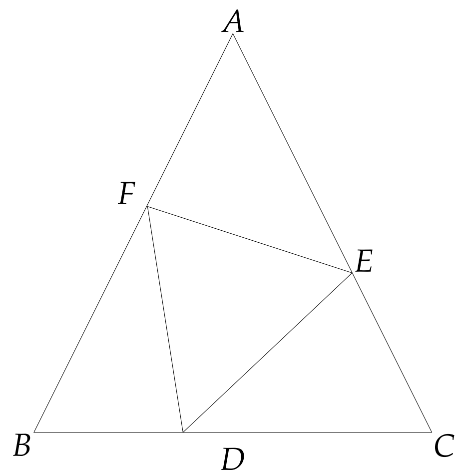
9) The equation $3 \cdot 2^m + 4 = n^2$ has three positive integer solution pairs, namely $(m, n) = \{(2, 4), (5, 10), (6, 14)\}$. The equation $3 \cdot 2^m + 16 = n^2$ also has three positive integer solution pairs, $(4, 8)$ and two others.

   Find three positive integer solution pairs to $3 \cdot 2^m + 2^{2008} = n^2$.

10) A swimming pool is built with the surface area in the shape of a circle with diameter 60 feet. Going from east to west, the depth drops linearly from 3 feet at the shallow end to 15 feet at the deep end but it does not vary along the north-south direction. Find the volume of the swimming pool.

Solutions

1)



Since a straight line is 180° and the angles of an equilateral triangle are 60°, we have $\angle BFD = 180^\circ - 60^\circ - 82^\circ = 38^\circ$ and $\angle FEA = 180^\circ - 60^\circ - 86^\circ = 34^\circ$.

Since the angles in a triangle sum to 180°, we have

$$\angle BAC = \angle FAE = 180^\circ - 82^\circ - 34^\circ = 64^\circ.$$

Since $\triangle ABC$ is isosceles,

$$\angle ABC = \frac{180^\circ - 64^\circ}{2} = 58^\circ.$$

Since the angles in a triangle sum to 180°, we have

$$\angle BDF = 180^\circ - 58^\circ - 38^\circ = 84^\circ.$$

Alternative Solution: By the exterior angle theorem, we have

$$(1)\ \angle AFE + 60^\circ = \angle ABC + \angle BDF \qquad \text{and} \qquad (2)\ \angle BDF + 60^\circ = \angle CED + \angle ACB.$$

But $\triangle ABC$ is isosceles, so $\angle ABC = \angle ACB$. Subtract (2) from (1) and rearrange to get

$$\angle BDF = \frac{\angle CED + \angle AFE}{2} = \frac{86^\circ + 82^\circ}{2} = 84^\circ.$$

2) Since MN is parallel to BC, we have $|AN| = |AM|$; denote this value by x. The area of $\triangle ABC$ is $\frac{1}{2}$. Since MN halves this area, the area of $\triangle AMN$ is $\frac{1}{4}$. So we have

$$\frac{x \cdot x}{2} = \frac{1}{4} \implies x^2 = \frac{1}{2} \implies x = \frac{1}{\sqrt{2}}$$

Therefore M is $\frac{1}{\sqrt{2}}$ units above the x-axis. That is, $M = \left(0, \frac{1}{\sqrt{2}}\right)$.

3) Let M be the number of men who attended the reunion. Then $\frac{3M}{4}$ women attended, so the total number of guests was $M + \frac{3M}{4} = \frac{7M}{4}$.

We are told half, or $\frac{M}{2}$, of the men spoke French. Since one third of the women did not speak French, two thirds of the women did speak French; that is, $\frac{2}{3} \cdot \frac{3M}{4} = \frac{M}{2}$ women spoke French. Thus the total number of guests who spoke French was $\frac{M}{2} + \frac{M}{2} = M$.

Therefore the desired ratio is $M : \frac{7M}{4}$, which is $4 : 7$ in lowest terms or $\frac{4}{7}$.

4) The houses numbered 1 through 99, inclusive, use a total of $9 \cdot 1 + 90 \cdot 2 = 189$ numerals, so if there are n three-digit house numbers in the subdivision then the total number of numerals used will be $189 + 3n$. Set $189 + 3n = 999$ to get $n = 270$. Hence there are $99 + 270 = 369$ houses in the subdivision.

5) Let A be the area of any of the shaded regions. Then the area of the smallest circle is $4A$, the area of the medium circle is $8A$, and the area of the large circle is $12A$. Since the radius of the smallest circle is 1, this circle has area π. Thus

$$4A = \pi \implies A = \frac{\pi}{4}$$

Therefore the middle circle has area $8 \cdot \frac{\pi}{4} = 2\pi$ and radius $\sqrt{2}$, while the large circle has area $12 \cdot \frac{\pi}{4} = 3\pi$ and radius $\sqrt{3}$. So the product of the radii is $1 \cdot \sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

Alternative Solution: The ratio of the areas of the three circles is $4 : 8 : 12$, or $1 : 2 : 3$. Thus their radii are in ratio $1 : \sqrt{2} : \sqrt{3}$. In fact, since the smallest circle is known to have radius 1, the radii of the circles are exactly $1, \sqrt{2}$, and $\sqrt{3}$. Hence their product is $\sqrt{6}$. (Here we have applied the useful observation that if two congruent figures have linear dimensions in ratio $\alpha : \beta$, then their areas are in ratio $\alpha^2 : \beta^2$.)

6) There are four possibilities for $f(f(n))$, depending on the value of n. These are:

(a) $\dfrac{n}{9}$ (b) $2\left(\dfrac{n}{3}\right) + 1$ (c) $\dfrac{2n+1}{3}$ (d) $2(2n+1)+1$.

We begin by ignoring the conditions on n that lead to each of these values for $f(f(n))$, and simply search in each case for n where it is *possible* that $f(f(n)) = n$.

(a) Here $f(f(n)) = n \iff \dfrac{n}{9} = n \iff n = 0$, impossible since n is a positive integer.

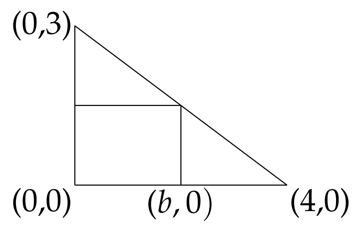
(b) In this case, $f(f(n)) = n \iff 2\left(\dfrac{n}{3}\right) + 1 = n \iff n = 3$.

(c) Here $f(f(n)) = n \iff \dfrac{2n+1}{3} = n \iff n = 1$.

(d) In this case $f(f(n)) = n \iff 2(2n+1)+1 = n$. But $2(2n+1)+1 > n$ for all $n \geq 1$, so $f(f(n)) = n$ is impossible here.

Therefore the only two possibilities are $n = 1$ and $n = 3$. A quick check shows that indeed $f(f(1)) = 1$ and $f(f(3)) = 3$, so there are exactly two positive integers with the desired property.

7) Draw the triangle in the (x, y)-plane.



The line connecting $(4, 0)$ and $(0, 3)$ has equation $y = -\frac{3}{4}x + 3$, so the upper-right corner of the inscribed rectangle has coordinates $(b, -\frac{3}{4}b + 3)$ for some b in the range $0 \leq b \leq 4$. The area of the rectangle is therefore

$$A = b\left(-\tfrac{3}{4}b + 3\right) = -\tfrac{3}{4}b^2 + 3b = -\tfrac{3}{4}(b - 2)^2 + 3.$$

Clearly A is maximized when $b = 2$, with maximal value $A = 3$.

8) Consider the following two events:

(a) Alice tosses more heads than Bridget.

(b) Alice tosses more tails than Bridget.

Observe that one of these events is sure to happen. In fact, *exactly* one of the events is certain to occur, since Alice has one more coin than Bridget and therefore cannot toss both more heads *and* more tails than Bridget. Moreover, both these events are equally likely since a heads and tails occur with probability $\frac{1}{2}$. Therefore the probability that Alice tosses more heads than Bridget is $\frac{1}{2}$.

Note: The same argument shows that whenever Alice has one more coin than Bridget, there is a 50–50 chance she will toss more heads.

Alternative Solution: Suppose Alice has a heads and Bridget has b heads after the first 8 coins have been tossed. There are three mutually exclusive possibilities: (1) $a > b$, (2) $a = b$, and (3) $a < b$. Let P_A, P_T and P_B, respectively, be the probability of these events. Then $P_A = P_B$ and $P_T = 1 - (P_A + P_B)$, so that $P_A = \frac{1}{2}(1 - P_T)$.

Consider the toss of the 9th coin. Let us say Alice *wins* if she ends up with more heads than Bridget. In case (1), any outcome on the 9th toss leads to Alice winning. In case (2), Alice wins only if the 9th toss is a head. Clearly Alice cannot win in case (3).

Thus the probability of Alice winning is $P_A + \frac{1}{2}P_T = \frac{1}{2}(1 - P_T) + \frac{1}{2}P_T = \frac{1}{2}$.

9) The key observation is that

$$3 \cdot 2^m + 4 = n^2 \implies 2^{2006}(3 \cdot 2^m + 2^2) = 2^{2006} n^2 \implies 3 \cdot 2^{m+2006} + 2^{2008} = (2^{1003} n)^2$$

So if (m, n) is a solution of $3 \cdot 2^m + 4 = n^2$ then $(\bar{m}, \bar{n}) = (m + 2006, 2^{1003} n)$ is a solution of $3 \cdot 2^{\bar{m}} + 2^{2008} = \bar{n}^2$. Since $(m, n) = (2, 4)$, $(5, 10)$, or $(6, 14)$ are solutions to the former equation, we find that $(\bar{m}, \bar{n}) = (2008, 2^{1005})$, $(2011, 5 \cdot 2^{1004})$, and $(2012, 7 \cdot 2^{1004})$ are three solutions to the latter equation.

Note: In fact, for any integer $k \geq 1$, we can show that there are *exactly* 3 positive integer solutions (m, n) to the equation

$$3 \cdot 2^m + 2^{2k} = n^2, \qquad (*)$$

namely, $(m, n) = (2k, 2^{k+1})$, $(2k + 3, 5 \cdot 2^k)$, and $(2k + 4, 7 \cdot 2^k)$.

To prove this claim, suppose (m, n) satisfies $(*)$ and consider the following three cases: (1) $m < 2k$, (2), $m = 2k$, and (3) $m > 2k$.

If $m < 2k$, then we have $2^m(3 + 2^{2k-m}) = n^2$. Since $3 + 2^{2k-m}$ is odd, 2^m is the highest power of 2 dividing n^2. Hence m is even, say $m = 2l$ for some $l \geq 1$, and we have $3 + 4^{k-l} = (\frac{n}{2^l})^2$. But this implies that we have a perfect square that leaves a remainder of 3 upon division by 4, which is impossible. So there are no solutions (m, n) with $m < 2k$.

If $m = 2k$, then dividing $(*)$ by 2^m gives $3 + 1 = (\frac{n}{2^k})^2$, so $n = 2^{k+1}$. So we get the unique solution $(m, n) = (2k, 2^{k+1})$ in this case.

Finally, if $m > 2k$, divide $(*)$ by 2^{2k} to get $3 \cdot 2^{m-2k} + 1 = (\frac{n}{2^k})^2$, and note that since the LHS is odd, the RHS must also be odd. Hence $\frac{n}{2^k} = 2j - 1$ for some $j \geq 1$, and we have $3 \cdot 2^{m-2k} = (2j-1)^2 - 1 = 4j(j-1)$. Since exactly one of j and $j - 1$ must be odd, we have $j = 3$ or $j - 1 = 3$. In the first case, we get $3 \cdot 2^{m-2k} = 4j(j-1) = 24$ so $m - 2k = 3$ and hence $n = 2^k(2j-1) = 5 \cdot 2^k$. In the latter case, get $3 \cdot 2^{m-2k} = 4j(j-1) = 48$ so $m - 2k = 4$ and $n = 2^k(2j-1) = 7 \cdot 2^k$. Thus we have the solutions $(m, n) = (2k+3, 5 \cdot 2^k)$ and $(m, n) = (2k + 4, 7 \cdot 2^k)$ with $m > 2k$.

10) Take two copies of such a pool, invert one, and place it on top of the other to form a cylinder of height $h = 3 + 15 = 18$ and radius $r = \frac{1}{2} \cdot 60 = 30$. The cylinder has volume

$$\pi r^2 h = 18 \cdot 30^2 \cdot \pi = 16200\pi$$

The volume of the swimming pool is half that of the cylinder, namely 8100π cubic feet.