Each question is worth ten points.

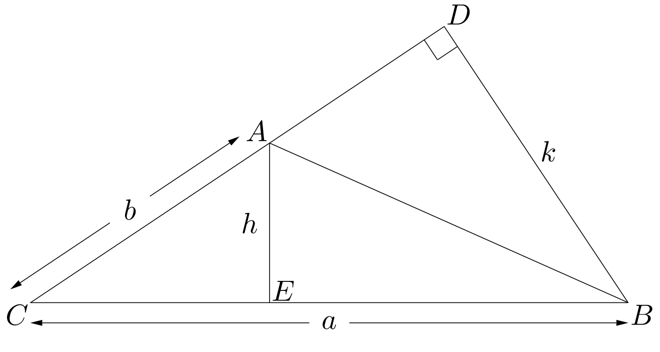
I. A function f goes from the positive integers (integers strictly greater than zero, i.e. 1,2,3, etc.) to the positive integers such that:

   (i) $f(2) = 2$;
   (ii) $f(n) > 0$ for every positive integer n;
   (iii) $f(n+1) > f(n)$ for every positive integer n;
   (iv) $f(nm) = f(n)f(m)$ for all positive integers n and m.

   Answer the following questions:

   (a) (1 point) Show that $f(1) = 1$.
   (b) (3 points) Show $f(4) = 4$ and $f(8) = 8$. Show that $f(2^k) = 2^k$ for every positive integer k.
   (c) (6 points) Explain why $f(n) = n$ for every positive integer n (Hint: look at $f(3)$).

II. Consider triangle $\triangle ABC$ with $\angle CAB$ obtuse and with altitudes of length h and k, as shown in the diagram below. All other variables are as in the diagram below.



   (a) (1 point) Show $\triangle AEC$ and $\triangle BDC$ are similar.
   (b) (1 point) Using (a) or otherwise, show $ah = kb$.
   (c) (6 points) Show $b + k \leq a + h$.
   (d) (2 points) Are there values for a, b, h, and k so that $b + k = a + h$? If so, list. If not, explain why.

Solutions

PLEASE NOTE: These are sample solutions and not the only solutions we would accept.

I. (a) Consider

$$\begin{aligned} f(1) &= f(1\cdot 1) && \\ f(1) &= f(1)\cdot f(1) && \text{by (iv)} \\ 1 &= f(1) && \text{dividing both sides by } f(1) \end{aligned}$$

Note that we can divide both sides by $f(1)$ since (ii) gives that $f(1) \neq 0$, ensuring that division by zero is not taking place.

An alternate solution is to note that (i) gives $f(2) = 2$ and using (ii) and (iii), we have:

$$0 < f(1) < f(2) = 2.$$

Since $f(1)$ must be a positive integer, this means that $f(1)$ is a positive integer between 0 and 2. Only one such integer, 1, exists, so $f(1) = 1$.

(b) Suppose we have $f(2^k)$. Then

$$\begin{aligned} f(2^k) &= f(2\cdot 2^{k-1}) && \\ &= f(2)\cdot f(2^{k-1}) && \text{by (iv)} \\ &= 2\cdot f(2^{k-1}) && \text{by (i)} \\ &= 2\cdot f(2\cdot 2^{k-2}) && \\ &= 2\cdot f(2)\cdot f(2^{k-2}) && \text{by (iv)} \\ &= 2^2\cdot f(2^{k-2}) && \text{by (i)} \\ &\vdots && \\ &= 2^{k-2}\cdot f(2\cdot 2) && \\ &= 2^{k-2}\cdot f(2)\cdot f(2) && \text{by (iv)} \\ &= 2^k && \text{by (i).} \end{aligned}$$

That is, at each step, we can separate 2^m and $2\cdot 2^{m-1}$ and apply (iv) so that $f(2\cdot 2^{m-1}) = f(2)\cdot f(2^{m-1})$. Since (i) gives $f(2) = 2$, we have $f(2^m) = 2\cdot f(2^{m-1})$. We then repeat the process again. Since we began with 2^k, the process repeats k times, and so $f(2^k) = 2^k$.

(This can also be formally shown by induction, although it is not required for full marks. Also showing the general case first and then saying that is shows $f(4) = 4$, $f(8) = 8$, and $f(16) = 16$ is acceptable.)

(c) In general, consider the integers between 2^k and 2^{k+1} inclusive. Then, by (ii), have

$$f(2^k) < f(2^k+1) < \ldots < f(2^k+[2^k-2]) < f(2^k+[2^k-1]) < f(2^{k+1}).$$

By (b), we have $f(2^k) = 2^k$ and $f(2^{k+1}) = 2^{k+1}$, so our inequalities become:

$$2^k < f(2^k+1) < \ldots < f(2^k+[2^k-2]) < f(2^k+[2^k-1]) < 2^{k+1}.$$

Now, there are $2^k - 1$ distinct integers between 2^k and 2^{k+1} exclusive, and note that $f(2^k+1)$, $f(2^k+2)$, …, $f(2^k+[2^k-2]), f(2^k+[2^k-1])$ forms a list of $2^k - 1$ distinct integers between 2^k and 2^{k+1} exclusive. Since this list is strictly increasing and the only list of strictly increasing integers between 2^k and 2^{k+1} exclusive is 2^k+1, 2^k+2, …, $2^k+[2^k-2]$, $2^k+[2^k-1]$, this means that $f(t) = t$ for $t \in \{2^k+1, 2^k+2, \ldots, 2^k+[2^k-1]\}$.

Therefore $f(n) = n$ for any n positive integer with $2^k < n < 2^{k+1}$. By (b) showed that $f(2^u) = 2^u$ for any u positive integer. Therefore, we have shown that for any positive integer, $f(n) = n$.

II. (a) Since h is an altitude $\angle AEC = 90°$. Since both triangles have one angle in common ($\angle C$) and both have one 90° angle ($\angle AEC$ and $\angle CDB$ respectively), by AAA, the two triangles are similar.

(b) Since the two triangles are similar, the ratio of their sides is equal. Since $AC \sim CB$ and $DB \sim AE$, have

$$\frac{AE}{AC} = \frac{DB}{CB}$$
$$\frac{h}{b} = \frac{k}{a}$$
$$ah = bk$$

(c) By Pythagorean Theorem, have $CD^2 + k^2 = a^2$. But $b^2 < CD^2$ so

$$b^2 + k^2 < CD^2 + k^2 = a^2.$$

Since $h^2 > 0$, have

$$b^2 + k^2 < CD^2 + k^2 = a^2 < a^2 + h^2.$$

Therefore $b^2 + k^2 < a^2 + h^2$. By (b), know that $ah = kb$, so $2ah = 2kb$. Adding this to both sides of the inequality, we obtain:

$$b^2 + k^2 + 2bk < a^2 + h^2 + 2ah,$$

which is the same as

$$(b+k)^2 < (a+h)^2.$$

Since both a, b, h, and k are positive, we obtain $b + k < a + h$, so $b + k \leq a + h$.

(d) In the work for (c), all the inequalities were strict. That is, $b + k < a + h$. Therefore, there are no values of a, b, h, and k such that the inequality of (c) holds with equality.