October 2008 Questions

1) The factorial $x!$ is defined for a positive integer x by the following

$$x! = x(x-1)(x-2)\cdots(3)(2)(1)$$

For example, $4! = (4)(3)(2)(1) = 24$.

(a) If $m \geq 3$, show that 3 does not divide $m! - 2$.

(b) Find all possible values of m, n such that $n! + 1 = (m! - 1)^2$.

Proof. (a) Know that for $m \geq 3$, 3 divides $m!$. Suppose 3 does divide $m! - 2$. Then there exists some $n \in \mathbb{N}$ such that $3n = m! - 2$. Rearranging, we get $m! - 3n = 2$. But $m! = 3t$ for $t = \dfrac{m!}{3}$ and t is a positive integer. Therefore, factoring out a 3 from the LHS gives $3(t - n) = 2$. So 3 divides the LHS, but 3 does not divide the RHS.

(b) Suppose $n! + 1 = (m! - 1)^2$.

If $m < 3$, then there are no possible values of n, given n is a positive integer, such that $n! + 1 = (m! - 1)^2$. Thus $m \geq 3$. We can also easily see that $n > m$.

Then

$$\begin{aligned} n! + 1 &= (m! - 1)^2 \\ n! &= m!(m! - 2) \\ n(n-1)(n-2)\cdots(m+1) &= m! - 2 \end{aligned}$$

We know that 3 divides $m!$, but we can see that 3 does not divide $m! - 2$. Thus 3 does not divide the LHS of the equation above. As such, the number of factors on the LHS is either 1 or 2, since every third number is divisble by 3.

Suppose there is only one factor on the LHS. Then $n = m + 1$ and $m + 1 = m! - 2$, or $m = m! - 3$. Since m divides m, and since m divides $m!$, this gives that m divides 3. But $m \geq 3$, so $m = 3$, and $n = 4$.

Suppose there are two factors on the LHS. Then $(m+1)(m+2) = m! - 2$. Then

$$\begin{aligned}(m+1)(m+2) &= m!-2\\ m^2+3m+2 &= m!-2\\ 3m &= m!-m^2-4\end{aligned}$$

But m divides $3m$, so m divides the $m!-m^2-4$, and since m divides $m!$ and m^2, we obtain that m divides 4. Since $m \geq 3$, we have that m must equal 4.

But then $3m = 3(4) = 12$ but $m! - m^2 - 4 = 24 - 16 - 4 = 4$. Thus there cannot be two factors on the LHS of the earlier equation.

Thus the only possible values are $(m, n) = (3, 4)$.

□

2) In the federal election, the ratio of male to female voters was $a : b$. Had c fewer men and d fewer women voted, the ratio would have been $e : f$. In terms of a, b, c, d, e, and f, determine the total number of votes cast.

Proof. Let m be the number of male voters and w be the number of female voters. Then $m + w$ is the total number of votes cast.

Since the ratio of male to female voters was $a : b$, we have

$$\frac{m}{w} = \frac{a}{b}.$$

Rearranging, we get

$$m = \frac{wa}{b}. \tag{1}$$

The second ratio in the question gives us

$$\frac{m-c}{w-d} = \frac{e}{f}. \tag{2}$$

Substituting (1) into (2) gives

$$\begin{aligned}
\frac{\left(\frac{wa}{b}\right) - c}{w - d} &= \frac{e}{f} \\
\frac{waf}{b} - cf &= we - de \\
waf - bcf &= wbe - bde \\
waf - wbe &= bcf - bde \\
w(af - be) &= b(cf - de) \\
w &= \frac{b(cf - de)}{af - be}
\end{aligned}$$

Substituting this value for w into (1) gives

$$\begin{aligned}
m &= \frac{wa}{b} \\
&= \frac{\left(\frac{b(cf-de)}{af-be}\right) a}{b} \\
&= \frac{a(cf - de)}{af - be}
\end{aligned}$$

Thus we have both m and w in terms of a, b, c, d, e, and f, and so

$$\begin{aligned}
m + w &= \frac{a(cf - de)}{af - be} + \frac{b(cf - de)}{af - be} \\
&= \frac{(a + b)(cf - de)}{af - be}
\end{aligned}$$

$\square$