September 2008 Questions

1) Let $S(n)$ be the sum of the first n terms of the following sequence:

$$0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, \ldots$$

(a) Show that

$$S(n) = \begin{cases} \dfrac{n^2}{4} & \text{if } n \text{ is even;} \\ \dfrac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

(b) Show $S(a+b) - S(a-b) = ab$ where a, b are positive integers with $a > b$.

Proof. (a) Suppose n is even. Then the first n terms of the sequence are:

$$0, 1, 1, 2, 2, 3, 3, \ldots, \left(\frac{n}{2} - 1\right), \left(\frac{n}{2} - 1\right), \frac{n}{2}.$$

Then

$$\begin{aligned} S(n) &= 0 + 1 + 1 + 2 + 2 + 3 + 3 + \ldots + \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2} \\ &= 0 + 1 + 2 + \ldots + \left(\frac{n}{2} - 1\right) + \\ &\qquad 1 + 2 + \ldots + \left(\frac{n}{2} - 1\right) + \frac{n}{2} \\ &= \frac{1}{2}\left(\frac{n}{2} - 1\right)\frac{n}{2} + \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right) \\ &= \frac{n}{4}\left(\frac{n}{2} - 1 + \frac{n}{2} + 1\right) \\ &= \frac{n^2}{4}. \end{aligned}$$

Suppose n is odd. Then the first n terms of the sequence are:

$$0, 1, 1, 2, 2, 3, 3, \ldots, \left(\frac{n}{2} - 1\right), \left(\frac{n}{2} - 1\right).$$

Then

$$\begin{aligned}
S(n) &= 0+1+1+2+2+3+3+\ldots+\frac{n-1}{2}+\frac{n-1}{2} \\
&= 0+1+2+3+\ldots+\frac{n-1}{2}+ \\
&\qquad 1+2+3+\ldots+\frac{n-1}{2} \\
&= 2\left(\frac{1}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right) \\
&= \frac{n^2-1}{4}
\end{aligned}$$

(b) Consider $a+b$ and $a-b$. Clearly since a, b, positive integers, $a+b > a-b$ and $(a+b)-(a-b)=2b$. Therefore $a+b$ and $a-b$ differ by an even number. That is, either $a+b$ and $a-b$ are even or $a+b$ and $a-b$ are odd.

Suppose $a+b$ and $a-b$ are even. Then

$$\begin{aligned}
f(a+b)-f(a-b) &= \frac{(a+b)^2}{4}-\frac{(a-b)^2}{4} \\
&= \frac{1}{4}(a^2+2ab+b^2-(a^2-2ab+b^2)) \\
&= \frac{1}{4}(4ab) \\
&= ab.
\end{aligned}$$

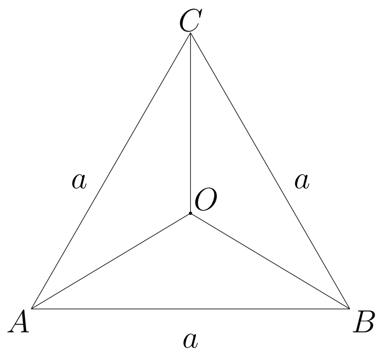
Suppose $a+b$ and $a-b$ are both odd. Then

$$\begin{aligned}
f(a+b)-f(a-b) &= \frac{(a+b)^2-1}{4}-\frac{(a-b)^2-1}{4} \\
&= \frac{1}{4}(a^2+2ab+b^2-1-(a^2-2ab+b^2-1)) \\
&= \frac{1}{4}(4ab) \\
&= ab.
\end{aligned}$$

□

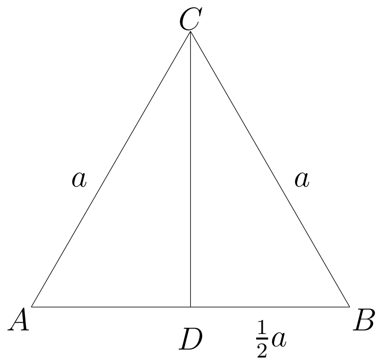
2) An equilateral triangle of area $3\sqrt{3}$ is inscribed in a circle. What is the radius of the circle?

Proof.



Suppose the above triangle is the equilateral triangle inscribed in the circle, with O the centre of the circle, and OA, OB, and OC radii of the circle.

Examine just the triangle:



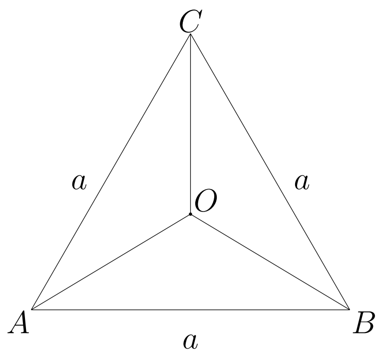
where D is the perpendicular bisector of AB. By Pythagorean theorem:

$$\begin{aligned} |CD|^2 + \left(\frac{1}{2}a\right)^2 &= a^2 \\ |CD|^2 &= \frac{3}{4}a^2 \\ |CD| &= \frac{\sqrt{3}}{2}a \end{aligned}$$

We are given that the area of the triangle is $3\sqrt{3}$. Therefore

$$\begin{aligned}
\text{Area} &= \frac{1}{2}bh \\
3\sqrt{3} &= \frac{1}{2}a\frac{\sqrt{3}}{2}a \\
3 &= \frac{1}{4}a^2 \\
12 &= a^2 \\
a &= 2\sqrt{3}
\end{aligned}$$

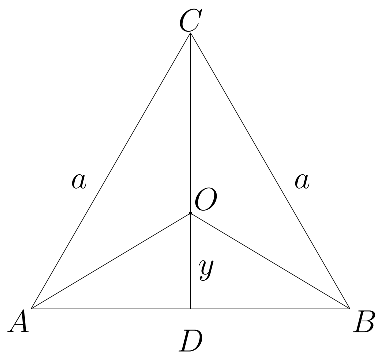
Draw in OA, OB, and OC again.



Then $\triangle ABO$, $\triangle ACO$, and $\triangle CBO$ are all the same triangle by Side-Side-Side. Therefore

$$\text{area of } \triangle ABC = 3 \cdot \text{area of } \triangle ABO$$

Let y be the height of triangle ABO.



Therefore

$$\begin{aligned}
\text{area of } \triangle ABC &= 3 \cdot \text{area of } \triangle ABO \\
3\sqrt{3} &= 3 \cdot \frac{1}{2} \cdot a \cdot y \\
3\sqrt{3} &= 3 \cdot \frac{1}{2} \cdot 2\sqrt{3} \cdot y \\
y &= 1
\end{aligned}$$

Since ABC is equilateral and the three inner triangles are the same, OD is a perpendicular bisector of AB. Therefore $\triangle ODA$ is a right angle triangle. By Pythagorean theorem

$$\begin{aligned}
|OA| &= \sqrt{y^2 + \left(\frac{1}{2}a\right)^2} \\
&= \sqrt{1^2 + \left(\frac{1}{2} \cdot 2\sqrt{3}\right)^2} \\
&= \sqrt{1 + (\sqrt{3})^2} \\
&= \sqrt{1+3} \\
&= \sqrt{4} \\
&= 2
\end{aligned}$$

$\square$