Nova Scotia Math League

2011–2012

Game Two

SOLUTIONS

Team Questions Solutions

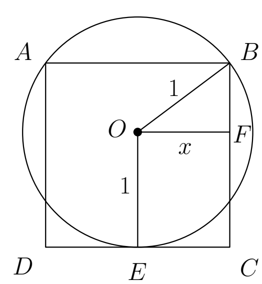
1. When Bob departs, Alan is 60 km in the lead. Bob drives at $100 - 60 = 40$ km/h relative to Alan, so they will meet $\frac{60}{40} = 1.5$ hours later. Since Bob leaves at 9am, he meets Alan at 10:30am.

2. The least element of S_{10} is $1 + (1 + 2 + 3 + \cdots + 9) = 46$, and it contains 10 elements. Thus $S_{10} = \{46, 47, 48, \ldots, 55\}$. The sum of these numbers is $\frac{10}{2}(46 + 55) = 505$.

3. Since $8 = 2^3$ we can rewrite $5^{80}8^{30}$ as $5^{80}2^{90} = (5 \cdot 2)^{80}2^{10} = 10^{80} \cdot 1024$. Expressed in decimal form, this number is 1024 followed by 80 zeros. So it contains a total of 84 digits.

4. There are $3 \cdot 2 = 6$ ways of finishing without any ties (choose the 1st place horse in 3 ways, then the 2nd place horse in 2 ways). There are 3 ways of finishing with a two-way tie for 1st place (since any one of 3 horses can then be in third place) and similarly there are 3 ways of finishing with a two-way tie for 2nd place. Finally there is one way in which all three horses can be tied for 1st place. So there are a total of $6 + 3 + 3 + 1 = 13$ possibilities.

5. In the figure below, O is the centre of the circle and OE and OF are perpendicular to CD and BC, respectively. Note that $|OB| = |OE| = |FC| = 1$. Let x be one-half the side of the square, so $|OF| = x$ and $|BC| = 2x$.



Then $|BF| = |BC| - |FC| = 2x - 1$, so Pythagorean theorem applies to $\triangle OBF$ to give

$$x^2 + (2x-1)^2 = 1^2 \quad \Longrightarrow \quad 5x^2 - 4x = 0 \quad \Longrightarrow \quad x(5x-4) = 0.$$

Thus $x = \frac{4}{5}$, and therefore the square has area $(2x)^2 = (\frac{8}{5})^2 = \frac{64}{25}$.

6. An $a \times b \times c$ box has volume is abc and surface area is $2(ab + bc + ca)$. So the "brute force" method of solving this problem would be to simply examine all possible integer triples (a, b, c) such that $abc = 120$, and choose the minimum value of $2(ab + bc + ca)$ from amongst them. However, a moment's reflection cuts down the work considerably.

If we weren't restricted to sides of integer lengths, observe that the minimum possible surface area would necessarily occur when all sides are equal — that is, when the box is a cube. (Think about symmetry or see the note below.) So it stands to reason that, in our case, the minimum should be achieved when a, b, c are as "equal" as possible under the condition $abc = 120$. The meaning of this is admittedly vague, but we can safely eliminate "skewed" tuples such as $(a, b, c) = (60, 2, 1)$ and $(20, 3, 2)$ from consideration.

To find possible values of a, b, c we first factor 120 completely as $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$. We now attempt "distribute" these factors amongst a, b, and c so as to keep these values approximately equal. The only real contender is $(4, 5, 6)$, or any of its rearrangements, which gives a surface area of $2(20 + 24 + 30) = 148$.

To check our intuition we can also test second and third-place finishers such as $(a, b, c) = (3, 5, 8)$ or $(2, 6, 10)$ to see that they give larger surface areas (158 and 184, respectively).

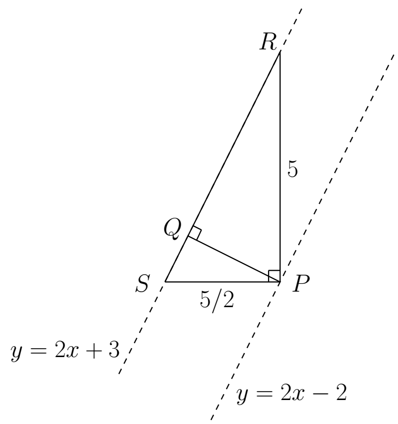
Note: The famous arithmetic-geometric mean inequality asserts that

$$\frac{ab + bc + ca}{3} \geq \sqrt[3]{ab \cdot bc \cdot ca} = (abc)^{2/3},$$

with equality if and only if $a = b = c$. Since an $a \times b \times c$ box has volume $V = abc$ and surface area $2(ab + bc + ca)$, this inequality assures us that any rectangular box of volume V has surface area *at least* $6V^{2/3}$. Moreover, the surface area can only be equal to this minimum value if all sides of the box are equal (i.e. it's a cube).

With $V = 120$, the minimum possible surface area of *any* rectangular box is therefore $6(120)^{2/3} \approx 146$. Compare this with the answer we obtained for boxes with integer sides, namely 148. These results are close enough to *guarantee* that 148 is indeed the minimum for integer length boxes, even if you didn't believe the argument about the sides being as "equal as possible".

7. The shortest distance between P and Q will occur whenever the line PQ is perpendicular to the two given lines. This situation is illustrated in the figure below, where we have also indicated points R and S on the "upper" line $y = 2x + 3$, directly above and to the left of P, respectively.



Since $(2x+3)-(2x-2)=5$ we have $|RP|=5$, and since RS has slope 2 we therefore get $|SP|=\frac{5}{2}$. Pythagorean theorem applied to $\triangle SPR$ then yields

$$|RS| = \sqrt{5^2 + \left(\frac{5}{2}\right)^2} = \frac{5\sqrt{5}}{2}.$$

But right triangles $\triangle SPR$ and $\triangle PQR$ are similar, so we have

$$\frac{|PQ|}{|RP|} = \frac{|SP|}{|RS|} \quad \Longrightarrow \quad \frac{|PQ|}{5} = \frac{5/2}{5\sqrt{5}/2} \quad \Longrightarrow \quad |PQ| = \sqrt{5}.$$

Note: The calculations are somewhat more straightforward if we simply note that $|SP|:|PR| = 1:2$ (because RS has slope 2), and therefore $|SP|:|RS| = 1:\sqrt{5}$ (by Pythagorean theorem). Since $|RP|=5$, we then get $|PQ| = 5/\sqrt{5}$, using similar triangles as above.

The typical high school method of solving this problem is to choose a point on one line, then find the equation of the perpendicular through this point, and then intersect this perpendicular with the other line. This method is laborious, although it is more obvious how it extends to higher dimensions.

8. Imagine Bobby selecting the socks one at a time. There are $10 \cdot 9 = 90$ possible selections, of which $5 \cdot 4 = 20$ consist of two *blue* socks. Thus there are $90 - 20 = 70$ possible selections with at least one red sock. Of these, $5 \cdot 4 = 20$ consist of *two red* socks. So the desired probability is $\frac{20}{70} = \frac{2}{7}$.

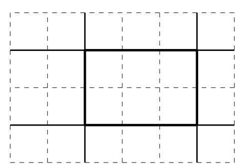
   Note: If Bobby instead tells you that the *second* sock that he chose was red, then strangely enough the probability that he chose two red socks now rises to $\frac{4}{9}$. Can you explain why?

9. Following the formula, we calculate the first few functions $f_n(x)$: This gives

$$\begin{aligned}
f_1(x) &= \frac{1}{1-f_0(x)} = \frac{1}{1-x} \\
f_2(x) &= \frac{1}{1-f_1(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x} \\
f_3(x) &= \frac{1}{1-f_2(x)} = \frac{1}{1-\frac{x-1}{x}} = x \\
f_4(x) &= \frac{1}{1-f_3(x)} = \frac{1}{1-x}.
\end{aligned}$$

At this point we stop our calculations because we recognize that the last calculation is simply a copy of the first! Of course, this occurred because $f_3(x)$ happened to be the same as $f_0(x)$. From now on, the sequence is destined to repeat itself, with a period of length 3. That is, $f_0, f_3, f_6, \ldots$ will be the same, as will $f_1, f_4, f_7, \ldots$ and $f_2, f_5, f_8, \ldots$. Since $2012 = 3 \cdot 67 + 2$, it follows that $f_{2012}(x) = f_2(x) = \frac{x-1}{x}$. Thus $f_{2012}(2012) = \frac{2011}{2012}$.

10. Every rectangle is determined by its four bounding lines — two vertical and two horizontal — as indicated below.



There are $\binom{7}{2} = 21$ choices for the two vertical lines and $\binom{5}{2} = 10$ choices for the horizontal lines, resulting in a total of $21 \cdot 10 = 210$ possible rectangles.

Pairs Relay Solutions

P-A. There are 5 even digits $(0,2,4,6,8)$ and 5 odd digits $(1,3,5,7,9)$. We want the number of pairs of digits XY such that Y is odd and $X \neq Y$. If X is even then Y can be any odd digit, resulting in $5 \cdot 5 = 25$ possibilities. If X is odd then Y can be only one of 4 remaining odd digits, yielding $5 \cdot 4 = 20$ possibilities. Altogether, there are $\mathtt{A} = 25 + 20 = 45$ possible numbers.

P-B. Since $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$ we have $10^2 = \mathtt{B} + 2\mathtt{A}$. With $\mathtt{A} = 45$ we get $\mathtt{B} = 10$.

P-C. The lines $y = 4x + 4$ and $y = \mathtt{B}x + \mathtt{B}$ each pass through $(x, y) = (-1, 0)$, and their y-intercepts are $y = 4$ and $y = \mathtt{B}$ respectively. Thus the triangle bounded by them and the y-axis has base (along the y-axis) of length $|\mathtt{B} - 4|$ and height 1. It's area is therefore $\mathtt{C} = \frac{1}{2}|\mathtt{B} - 4|$. With $\mathtt{B} = 10$ we get $\mathtt{C} = 3$.

P-D. This problem is readily solved by inspection by counting the possibilities in an organized fashion. But here is a systematic method:

The key observation is that to create a total of \$C, one requires ℓ loonies and q quarters, where $0 \leq \ell \leq \mathtt{C}$ and q is an *even* integer in the range $0 \leq q \leq 4(\mathtt{C} - \ell)$. Moreover, if ℓ and q that satisfy these conditions, then some number of dimes can be added to ℓ loonies and q quarters to make the total value of the coins be exactly \$C. Our goal is therefore to count valid pairs (ℓ, q).

Since there are $2(\mathtt{C} - \ell) + 1$ even values of q in the range $0 \leq q \leq 4(\mathtt{C} - \ell)$, the answer to the problem is obtained by summing $2(\mathtt{C} - \ell) + 1$ over all values of ℓ between 0 and $\mathtt{C}$, inclusive.

With $\mathtt{C} = 3$, this results in $7 + 5 + 3 + 1 = 16$.

Note: The general answer is $\mathtt{C}^2$, since this is the sum of the first $\mathtt{C}$ odd positive integers.

Individual Relay Solutions

I-A. Jack is $1.4 \cdot 0.75 = 1.05$ times as heavy as Ed. That is, he is 5% heavier, so $\mathtt{A} = 5$.

I-B. Factor as $(x-2)(2x - \mathtt{A} - 1) = 0$. The roots of this equation are $x = 2$ and $x = \frac{1}{2}(\mathtt{A}+1)$.

With $\mathtt{A} = 5$, the desired sum is $\mathtt{B} = 2 + \frac{1}{2}(6) = 5$.

I-C. Solution Z contains a total of

$$3 \cdot \frac{7.5}{\mathtt{B}} + 2 \cdot \frac{11}{4} = \frac{90 + 22\mathtt{B}}{4\mathtt{B}}$$

grams of salt, dissolved in 5 litres of water. In one litre of Solution Z there would therefore be $\frac{1}{5}$ this amount of salt.

With $\mathtt{B} = 5$, this results in $\mathtt{C} = \frac{1}{5} \cdot \frac{1}{20}(90 + 22 \cdot 5) = 2$.

I-D. This can be evaluated without any trickery, simply by plugging in the given value of $\mathtt{C}$. But one can also simplify first:

$$\frac{2^{\mathtt{C}} - 2^{-\mathtt{C}}}{4^{\mathtt{C}} - 4^{-\mathtt{C}}} = \frac{2^{\mathtt{C}} - 2^{-\mathtt{C}}}{(2^{\mathtt{C}} - 2^{-\mathtt{C}})(2^{\mathtt{C}} + 2^{-\mathtt{C}})} = \frac{1}{2^{\mathtt{C}} + 2^{-\mathtt{C}}} = \frac{2^{\mathtt{C}}}{2^{2\mathtt{C}} + 1}.$$

The numerator of this fraction is a power of 2 and the denominator is odd, so it must be in lowest terms. Hence $\mathtt{D} = 2^{2\mathtt{C}} + 1$.

With $\mathtt{C} = 2$, we get $\mathtt{D} = 2^4 + 1 = 17$.