Nova Scotia Math League

2012–2013

Game Three

SOLUTIONS

Team Question Solutions

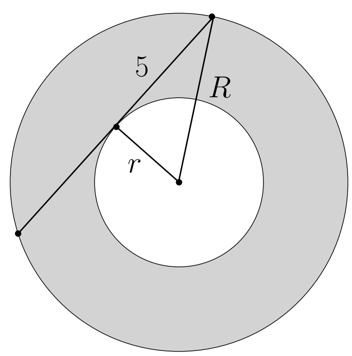
1. Using the recursion to generate the first few values of a_n yields the following.

n	0	1	2	3	4	5	6
a_n	1	2	5	10	17	26	37

   That's enough evidence to conjecture the general formula $a_n = n^2 + 1$, from which we get $a_{100} = 100^2 + 1 = 10001$.

   Note: It's a good exercise in mathematical induction to prove that the conjectured formula does indeed hold!

2. Let the radii of the large and small circles be R and r, respectively. Then the desired area is clearly $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. Draw a radius of the small circle to meet the chord perpendicularly at the point of tangency, as indicated in the diagram below. Note that this bisects the chord and creates a right triangle with legs r and R, and hypotenuse 5. Pythagorean theorem then gives $R^2 - r^2 = 25$, so that the desired area is 25π.



3. Let Jack and Zoë's speeds be j and z, respectively (measured in the same units). When skating in the same direction, Jack's speed relative to Zoë is $j - z$, whereas it is $j + z$ when skating in opposite directions. If they meet each other 9 times as frequently when skating in opposite directions, it must be the case that $j + z = 9(j - z)$. Rearranging this equality yields $8j = 10z$, so that $\frac{j}{z} = \frac{5}{4}$.

4. Since $27 = 3^3$, we have

$$\frac{3^{x^2+8}}{27^{2x+1}} = \frac{3^{x^2+8}}{3^{6x+3}} = 3^{x^2-6x+5} = 3^{(x-3)^2-4}.$$

   The quadratic in the exponent takes on a minimum value of -4 when $x = 3$, so the power takes on a minimum value of $3^{-4} = \frac{1}{81}$ at $x = 3$.

5. Note that $\triangle ADC$ and $\triangle BEC$ are both right angled triangles which contain $\angle C$. They are therefore similar, so we have

$$\begin{aligned} \frac{|CB|}{|EC|} = \frac{|AC|}{|CD|} \quad &\Longrightarrow \quad \frac{|CD|+|DB|}{|EC|} = \frac{|AE|+|EC|}{|CD|} \\ &\Longrightarrow \quad \frac{3+|DB|}{2} = \frac{2+4}{3} \\ &\Longrightarrow \quad |DB| = 5. \end{aligned}$$

6. Notice that $\lfloor \sqrt{x} \rfloor = n$ precisely when $n^2 \le x < (n+1)^2$, and there are $(n+1)^2 - n^2 = 2n+1$ integer values of x in this range. That is, we have

$$\left\lfloor \sqrt{n^2} \right\rfloor + \left\lfloor \sqrt{n^2+1} \right\rfloor + \cdots + \left\lfloor \sqrt{(n+1)^2-1} \right\rfloor = n(2n+1)$$

for all integers n. We obtain $\lfloor 1 \rfloor + \lfloor 2 \rfloor + \cdots + \lfloor 99 \rfloor$ by summing this expression from $n = 1$ to $n = 9$. Thus we have

$$\begin{aligned} \lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \cdots + \lfloor \sqrt{100} \rfloor &= \lfloor 100 \rfloor + \sum_{n=1}^{9} n(2n+1) \\ &= 10 + 1(3) + 2(5) + 3(7) + \cdots + 9(19) \\ &= 625. \end{aligned}$$

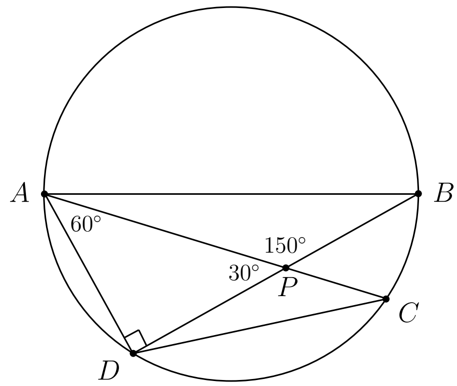
Note: The summation above can be computed quickly using the well known formulae $\sum_{n=1}^{m} = \frac{1}{2}m(m+1)$ and $\sum_{n=1}^{m} n^2 = \frac{1}{6}m(m+1)(2m+1)$, as follows:

$$\sum_{n=1}^{9} n(2n+1) = 2\sum_{n=1}^{9} n^2 + \sum_{n=1}^{9} n = 2 \cdot \frac{9(9+1)(2\cdot 9+1)}{6} + \frac{9(9+1)}{2} = 625$$

7. The vertices of the hexagon lie on a circle. Since a triangle inscribed in a circle is right-angled if and only if one of its sides is a diameter, the three chosen vertices must be comprised of a diametrically opposite pair along with any additional vertex. There are 3 diagonal pairs, and for each such pair there are 4 choices of the additional vertex, yielding $3 \cdot 4 = 12$ right triangles. There are $\binom{6}{3} = 20$ choices of three vertices altogether, so the desired probability is $\frac{12}{20} = \frac{3}{5}$.

   Note: It is a good exercise to work out the answer of this problem in the general case where the vertices are selected from a $2n$-gon. And what is the probability of the triangle being acute? Obtuse?

8. First observe that $\angle ABP = \angle ACD$ since both angles are subtended by arc AD. Since $\angle APB = \angle DPC$, it follows that triangles $\triangle APB$ and $\triangle DPC$ are similar. Now draw line AD and observe that $\triangle ABD$ is inscribed in a circle with side AB a diameter.



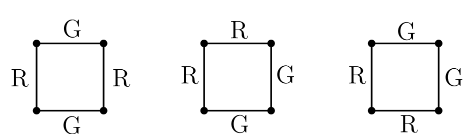
It follows that $\angle ADB = 90^\circ$. But $\angle APB = 150^\circ$ implies $\angle APD = 30^\circ$, so $\triangle PAD$ is a 30-60-90 triangle and hence $|AP| : |PD| = 2 : \sqrt{3}$. Since the ratio of similarity between $\triangle APB$ and $\triangle DPC$ is $2 : \sqrt{3}$, the ratio of their areas is $4 : 3$.

9. Notice that

$$\begin{aligned} S &= (10-1) + (10^2-1) + (10^3-1) + \cdots + (10^{2013}-1) \\ &= (10 + 10^2 + 10^3 + \cdots + 10^{2013}) - 2013 \\ &= (10^5 + 10^6 + 10^6 + \cdots + 10^{2013}) + 11110 - 2013 \\ &= \overbrace{11111\cdots111}^{\text{2009 ones}}00000 + 9097 \\ &= \overbrace{11111\cdots111}^{\text{2009 ones}}09097. \end{aligned}$$

So the sum of the digits of S is $2009 + 9 + 9 + 7 = 2034$

10. The key observation is that, once one side of a square has been coloured, there are only 3 ways to complete the colouring of the square. For instance, if the left side of a square is red, then the possible colourings are



Let us colour the figure from left-to-right. The first (leftmost) vertical segment can be coloured in 2 ways. For each choice, the colouring of the first square can be completed in 3 ways. Any such colouring fixes the colour of the second vertical segment, so the colouring of the second square can again be completed in 3 possible ways. The same holds for the third square, and we have a total of $2 \cdot 3 \cdot 3 \cdot 3 = 54$ possible colourings.

Note: Of course, the argument used above generalizes to yield $2 \cdot 3^n$ colourings of the segments of a $1 \times m$ grid. For a challenge, try to use similar logic to find the number of colourings of an $1 \times m$ grid, this time using *three* colours. (Again, each square should have two sides of one colour and two sides of another.) Now see if you can extend your result to find the number of colourings of a $n \times m$ grid with 3 colours.

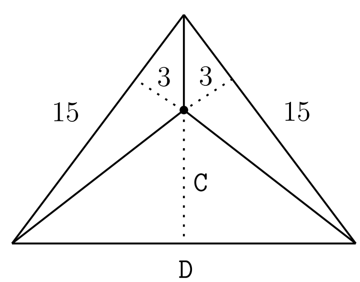
Pairs Relay Solutions

P-A. Since the given expression is symmetric, we only need to evaluate it with (x, y, z) equal to $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$. Doing so yields $0, 1, \frac{3}{2}$, and $\frac{3}{2}$. Thus there are $\mathtt{A} = 3$ distinct possible values of the expression.

P-B. The x-intercepts of the lines with slope 2 and 3 will be $10 - \mathtt{A}/2$ and $10 - \mathtt{A}/3$, respectively. The distance between the x-intercepts is the difference between these quantities, namely $\mathtt{A}/6$. With $\mathtt{A} = 3$ we get $\mathtt{B} = \frac{3}{6} = \frac{1}{2}$.

P-C. Note that $4^k = 2^{2k}$. Since 2 is prime, the number 2^{2k} has exactly $2k + 1$ positive divisors, namely $1, 2, 2^2, 2^3, \ldots, 2^{2k}$. With $\mathtt{B} = \frac{1}{2}$ we get $k = 3$ and hence $\mathtt{C} = 2(3) + 1 = 7$.

P-D. Draw lines from each vertex to P to separate the triangle into 3 smaller triangles, two with base 15 and altitude 3, and one with base $\mathtt{D}$ and altitude $\mathtt{C}$. (See the diagram below.) Summing the areas of each of these small triangles gives the area of the large triangle as $\frac{1}{2}(15)(3) + \frac{1}{2}(15)(3) + \frac{1}{2}\mathtt{DC}$. Thus we have $108 = 45 + \frac{1}{2}\mathtt{CD}$, or simply $\mathtt{D} = 126/\mathtt{C}$. With $\mathtt{C} = 7$ we get $\mathtt{D} = 18$.



Individual Relay Answer Key

I-A. We know $|AB| \cdot |AD| - |AE|^2 = 230$. Thus $(\mathtt{A}+10)(\mathtt{A}+5) - \mathtt{A}^2 = 230$, which gives $15\mathtt{A} + 50 = 230$, and hence $\mathtt{A} = 12$.

I-B. We have $\mathtt{A} = 2^k n$, where n is odd. Then $\mathtt{A}^{\mathtt{A}} = 2^{k\mathtt{A}} n^{\mathtt{A}}$, and the highest power of 2 dividing this number is $\mathtt{B} = k\mathtt{A}$. With $\mathtt{A} = 12 = 2^2 \cdot 3$ we have $k = 2$ and hence $\mathtt{B} = 2 \cdot 12 = 24$.

I-C. From the given equation we know $r + s = \sqrt{\mathtt{B}}$ and $rs = 2$, whence

$$\begin{aligned} \mathtt{C} = r^2 + s^2 &= (r+s)^2 - 2rs \\ &= (\sqrt{\mathtt{B}})^2 - 2(2) \\ &= \mathtt{B} - 4. \end{aligned}$$

With $\mathtt{B} = 24$ we have $\mathtt{C} = 20$.

I-D. The original cube has surface area $6\mathtt{C}^2$. After the tunnel is bored the surface area is $6\mathtt{C}^2 - 8 + 4 \cdot 2\mathtt{C}$, since we remove 4 squares from 2 faces but add the surface of the tunnel itself, which has 4 sides, each with area $2\mathtt{C}$. The difference between these quantities is $\mathtt{D} = 8\mathtt{C} - 8$. With $\mathtt{C} = 20$ we get $\mathtt{D} = 152$.