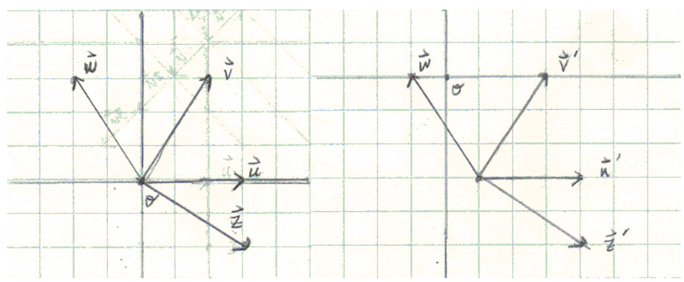
MATH 2030: ASSIGNMENT 1 SOLUTIONS

Geometry and Algebra of Vectors

Q.1: pg 16, q 1,2. For each vector draw the vector in standard position and with its tail at the point $(1, -3)$:

$$\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \ \mathbf{z} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$



Q.2: pg 17, 16. Simplify the vector expression, and indicate which properties from Theorem 0.6 are being used.

$$-3(\mathbf{u} - \mathbf{w}) + 2(\mathbf{u} + 2\mathbf{v}) + 3(\mathbf{w} - \mathbf{v})$$

A.2: To start we use property (5) and (7) to get

$$(-3\mathbf{u} + 3\mathbf{w}) + (2\mathbf{u} + 2\mathbf{v}) + (3\mathbf{w} - 3\mathbf{v})$$

then by applying property (2) and (4) twice we find

$$-\mathbf{u} + 4\mathbf{v} + 3\mathbf{w} + (3\mathbf{w} - 3\mathbf{v})$$
$$-\mathbf{u} + \mathbf{v} + 6\mathbf{w}$$

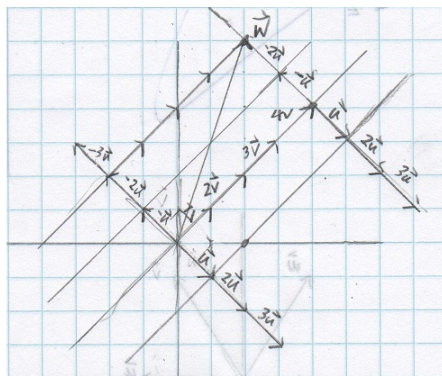
Q.3: pg 17, pg 17,18. Solve for the vector $\mathbf{w}$ in terms of $\mathbf{u}$ and $\mathbf{v}$:

- $\mathbf{w} - \mathbf{u} = 2(\mathbf{w} - 2\mathbf{u})$
- $\mathbf{w} + 2\mathbf{u} - \mathbf{v} = 3(\mathbf{w} + \mathbf{u}) - 2(2\mathbf{u} - 2\mathbf{v})$

A.3.

- Solving for $\mathbf{w}$ gives $\mathbf{w} = 3\mathbf{u}$.
- Here, $\mathbf{w} = -\frac{1}{2}(\mathbf{u} + \mathbf{v})$.

Q.4: pg 17, p 21. In $\mathbb{R}^2$ given two vectors $\mathbf{u}$, $\mathbf{v}$ such that $\mathbf{u} \neq c\mathbf{v}$ for all $c \in \mathbb{R}$, we may fill the coordinate plane with parallelograms with $\mathbf{u}$ and $\mathbf{v}$ as sides. In essence this construction gives a new coordinate system for the plane, e.g. consider $\mathbf{u}^t = [1, 0]$ and $\mathbf{v}^t = [0, 1]$ - these produce the standard coordinate axes.
Given $\mathbf{u}^t = [1, -1]$ and $\mathbf{v}^t = [1, 1]$, draw the standard coordinate axes on the same diagram as the axes relative to $\mathbf{u}$ and $\mathbf{v}$. Use these to find $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{w}^t = [2, 6]$.



Q.5: pg 17 q 44,46,48. Solve the given equation or indicate there is no solution

- $2x = 1$ in $\mathbb{Z}_3$
- $x + 3 = 2$ in $\mathbb{Z}_5$
- $2x = 1$ in $\mathbb{Z}_5$

A.5.

- Trying each element in $\mathbb{Z}_3$ we find $x = 2$ satisfies the equation as $2 \times 2 = 4 = 3 \times 1 + 1$, and so the remainder is 1.
- By exhaustion we find $x = 4$ satisfies the equation in $\mathbb{Z}_5$.
- Here, $x = 3$ is the only solution in $\mathbb{Z}_5$.

The dot product

Q.6: pg 29 q 4,12. Given $\mathbf{u}^t = [1.5, 0.4, -2.1]$ and $\mathbf{v}^t = [3.0, 5.2, -0.6]$ calculate $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{u}||$ and finally give the unit vector in the direction of $\mathbf{u}$.

A.6. Calculating $\mathbf{u} \cdot \mathbf{v} = (1.5)(3.0) + (0.4)(5.2) + (2.1)(0.6) \approx 4.5 + 2.1 + 1.3 = 7.9$. The magnitude of u, is simply $||\mathbf{u}|| = \sqrt{(1.5)^2 + (0.4)^2 + (2.1)^2} \approx \sqrt{6.8} = 2.6$. Using this we find that the unit vector in the $\mathbf{u}$-direction may be approxiated as

$$\mathbf{u}' = \frac{\mathbf{u}}{||\mathbf{u}||} \approx \frac{1}{2.6}[1.5, 0.4, -2.1]$$

Q.7: pg 29 q 20,26. Determine the angle between $\mathbf{u} = [5, 4, -3]$ and $\mathbf{v} = [1, -2, -1]$ is acute, obtuse or a right angle by calculating it explicitly.

A.7. Calculating $\mathbf{u} \cdot \mathbf{v} = 5 * 1 - 2 * 4 + 3 * 1 = 5 - 8 + 3 = 0$, we conclude the angle is a right angle, implying the two vectors are orthogonal.

Q.8: pg 29 q 36. An airplane heading due east has a velocity of 200 miles per hour. A wind is blowing from the north at 40 miles per hour. What is the resultant velocity of the plane?

A.8. The resultant velocity will be $\sqrt{200^2 + 40^2} = \sqrt{41600} \approx 204$ with an angle of θ from east with $tan\theta = \frac{40}{200} = 0.2$, i.e. $\theta \approx 0.2$ radians.

Q.9: pg 30 q 42. Find the projection of $\mathbf{v}$ onto $\mathbf{u}$ where $\mathbf{u}^t = [\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}]$ and $\mathbf{v}^t = [2, -2, 2]$.

A.9. Calculating $||\mathbf{u}|| = 1$, and $\mathbf{u} \cdot \mathbf{v} = \frac{4}{3} + \frac{4}{3} - \frac{2}{3} = \frac{6}{3} = 2$ we find that $\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} = 2$ and so

$$proj_{\mathbf{u}}(\mathbf{v}) = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ -\frac{2}{3} \end{bmatrix}$$

Q.10: pg 30 q 60. Suppose we know that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Does it follow that $\mathbf{v} = \mathbf{w}$? If it does, give a proof that is valid in $\mathbb{R}^n$; otherwise, give a *counterexample* (that is, a specific choice of vectors for which $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$).

A.10. As a counter-example, consider $\mathbf{u} = [1, 0, 0]$, $\mathbf{v} = [0, 1, 0]$ and $\mathbf{w} = [0, 0, 1]$; the dot product $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \cdot \mathbf{w} = 0$ however $\mathbf{v} \neq \mathbf{w}$.

Lines and Planes

Q.11: pg 44 q 6. Write the equation of the line passing through P with direction vector $\mathbf{d}$ in vector form and parametric form:

$$P = (-3, 1, 2), \ \mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

A.11. Taking the coordinate and producing the vector in standard position $\mathbf{p} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ we find that the equation of the line will be $\mathbf{x} = \mathbf{p} + s\mathbf{d}$.

Expanding this into components we find that the parametric form is

$$x = t - 3, \ y = 1, \ z = 2 + 5t.$$

Q.12: pg 44 q 12 . Give the vector equation of the line passing through $P = (4, -1, 3)$ and $Q = (2, 1, 3)$.

A.12. Calculating the direction vector as $\mathbf{d} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ the vector equation for this line is then $\mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

Q.13: pg 44 q 14 . Give the vector equation of the plane passing through $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$.

A.13. Calculating $\mathbf{d}_1 = \mathbf{q} - \mathbf{p} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{d}_2 = \mathbf{r} - \mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ we find that the vector equation is then $\mathbf{x} = \mathbf{p} + t\mathbf{d}_1 + s\mathbf{d}_2$.

Q.14: pg 45 q 20 . Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is perpendicular to the line with general equation $2x - 3y = 1$.

A.14. For a general equation of a line in $\mathbb{R}^2$, $ax + by = c$ the normal vector to this line will be $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ and so the vector equation of a line through P and perpendicular to the original line is then $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Q.15: pg 45 q 27 . Find the distance from the point $Q = (2, 2)$ and the line ℓ with the equation,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A.15. Taking the parametric equations for the line and solving for t in terms of x and substituting this into the y equation gives the general equation $x + y = 1$ thus using the standard distance formula for the plane

$$d(Q, \ell) = \frac{|aq_1 + bq_2 - c|}{\sqrt{a^2 + b^2}} = \frac{1 * 2 + 1 * 2 - 1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Q.16: pg 46 q 36 . Find the distance between the parallel lines:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A.16. Definining $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ as the position vectors for two points - one on each line. We may take their difference to produce a new vector connecting the two lines:

$$\mathbf{v} = \mathbf{p} - \mathbf{q} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Calculating the projection of this vector onto the direction vector we find

$$proj_{\mathbf{d}}(\mathbf{v}) = -\frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Subtracting this from $\mathbf{v}$ we find the required vector that measures the shortest distance between the lines, i.e. its magnitude will be the distance between the lines:

$$||\mathbf{v} - proj_{\mathbf{d}}(\mathbf{v})|| = \sqrt{\frac{14}{3}}.$$

Q.17: pg 46 q 43 . Find the acute angle between the planes with the equations:

$$x + y + z = 0,\ 2x + y - 2z = 0.$$

A.17. To calculate the angle between the planes, we note that for any plane in $\mathbb{R}^3$ there is a unique vector that is orthogonal to the plane, and so if we consider the angle between the *normal* vectors of the two planes we will have found the angle between the planes.

As the general equations are listed, we may immediately identify the normal vectors for each plane:

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{n}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Calculating the angle formula we find that the angle between these vectors satisfiy the identity:

$$cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1||||\mathbf{n}_2||} = \frac{1}{3\sqrt{3}}$$

Applying arccosine (that is, the *inverse function* for cosine on the interval $[0, \pi]$, θ may be determined.

Applications of Vectors

Q.18: pg 58 q 18,20 . A parity check code vector $\mathbf{v}$ is given, determine whether a single error could have occurred in the transmission of $\mathbf{v}$:

- $\mathbf{v} = [1, 1, 1, 0, 1, 1]$
- $\mathbf{v} = [1, 1, 0, 1, 0, 1, 1, 1]$

A.18.

- In this case there are 4 1s and so the check digit should be 0, since it is an even number of 1s. Hence an error was detected.
- There are 5 1s, and the check digit is 1, no error was detected.

Q.19: pg 58 q 24 . Consider the UPC $[0, 4, 6, 9, 5, 6, 1, 8, 2, 0, 1, 5]$

- Show that this UPC cannot be correct.
- Assuming that a single error was made and that the incorrect digit is the 6 in the third entry, find the correct UPC.

A.19.

- The sum is equivalent to the linear equation $3(6+9)+2 = 7$ mod 10, and so this is incorrect as the sum should be 0 mod 10.
- If we change 6 to 7 we find $3(7+9)+2 = 0$ mod 10, this gives a correct UPC.

Q.20: pg 58 q 30 .

- Prove that if a transposition error is made in the second and third entries of the UPC with
$$[0, 7, 4, 9, 2, 7, 0, 2, 0, 9, 4, 6]$$
the error will be detected.
- Show that there is a transposition involving two adjacent entries of the UPC in the first part that would not be detected.

A.20.

- Switching 4 and 7, we calculate the dot produce with the check vector,
$$3(0+7+2+0+0+4)+(4+9+7+2+9+6) = 3*3+7 = 9+7 = 6 mod 10$$
we conclude an error has been detected as this dot product must vanish.
- Notice that
$$3*4+9 = 1 = 1 mod 10, \ 3*9+4 = 1 mod 10$$
Hence if we switch the fourth and fifth entries no error would be detected.

References

[1] D. Poole, Linear Algebra: A modern introduction - 3rd Edition, Brooks/Cole (2012).