Math 1000 Practice Final Exam Solutions

Note: Final Exam is Wednesday, August 19, 2009 at 6:05pm in Dunn 302.

1. [3 marks] Consider $f(x) = \begin{cases} \frac{x^2+x-6}{x+3} & \text{if } x < -3 \\ \sqrt{x+4} & x \geq -3 \end{cases}$. Find the following limits:

(a) $\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{x+4} = \sqrt{1} = 1$

(b) $\lim_{x\to -3^-} f(x) \lim_{x\to -3^-} \dfrac{x^2+x-6}{x+3} = \lim_{x\to -3^-} \dfrac{(x+3)(x-2)}{x+3} = \lim_{x\to -3^-} x-2 = -5$

(c) $\lim_{x\to -3} f(x)$ does not exist

2. [6 marks] Find each of the following limits:

(a)

$$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \lim_{x\to 0} \frac{(x+9)-9}{x(\sqrt{x+9}+3)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+9}+3)} = \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{6}$$

(b)

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(4x)} = \lim_{x\to 0} \frac{1}{\sin(4x)} \cdot \frac{\sin(3x)}{\cos(3x)} \cdot \frac{3x}{3x} \cdot \frac{1/4x}{1/4x} = \lim_{x\to 0} = \frac{1}{\frac{\sin(4x)}{4x}} \frac{\sin(3x)}{3x} \frac{1}{\cos(3x)} \frac{3}{4} = \frac{3}{4}$$

3. [4 marks] (a) Give the limit definition of the derivative.

The derivative of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

(b) Use the **limit definition of derivative** to find $f'(2)$ if $f(x) = \sqrt{x+2}$.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2+h+2}-\sqrt{2+2}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4} \end{aligned}$$

4. [3 marks] (a) State the Mean Value Theorem.

Let f be a function that satisfies the following conditions: (i) f is continuous on the closed interval $[a, b]$ and (ii) f is differentiable on the open interval (a, b). Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let $f(x) = x^2$ on the interval $[-2, 1]$. Show that the Mean Value Theorem applies and then find all the values of c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since f is a polynomial, it is continuous on $[-2, 1]$ and it is differentiable on $(-2, 1)$. Thus, the Mean Value Theorem applies.

Differentiating $f'(x) = 2x$. By the Mean Value Theorem, we know that

$$\begin{aligned} f'(c) = 2c &= \frac{f(1) - f(-2)}{1 - (-2)} \\ &= \frac{(1)^2 - (-2)^2}{1 - (-2)} \\ &= \frac{1 - 4}{3} = -1 \\ c &= \frac{-1}{2} \end{aligned}$$

5. [12 marks] Differentiate the following functions:

(a) $f(x) = \arctan(x^2) + xe^{ax}$, where a is a constant

$$f'(x) = \frac{1}{1 + (x^2)^2}(2x) + e^{ax} + xe^{ax}(a) = \frac{2x}{1 + x^4} + e^{ax} + axe^{ax}$$

(b)

$$g(x) = \frac{2x^3 + x^2 + x + 1}{x^2 + 1}$$

$$\begin{aligned} g'(x) &= \frac{(x^2 + 1)(6x^2 + 2x + 1) - (2x)(2x^3 + x^2 + x + 1)}{(x^2 + 1)^2} \\ &= \frac{6x^4 + 2x^3 + x^2 + 6x^2 + 2x + 1 - (4x^4 + 2x^3 + 2x^2 + 2x)}{(x^2 + 1)^2} \\ &= \frac{2x^4 + 5x^2 + 1}{(x^2 + 1)^2} \end{aligned}$$

(c) $H(x) = \arcsin(\sqrt{1-x^2})$

$$\begin{aligned} H'(x) &= \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}}\frac{1}{2}(1-x^2)^{-1/2}(-2) \\ &= \frac{1}{\sqrt{1-(1-x^2)}}\frac{-1}{\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{x^2}}\frac{1}{\sqrt{1-x^2}} \\ &= \frac{1}{x\sqrt{1-x^2}} \end{aligned}$$

(d) $f(t) = \ln(\sin(\ln(t)))$

$$\begin{aligned} f'(t) &= \frac{1}{\sin(\ln(t))}\cos(\ln(t))\frac{1}{t} \\ &= \frac{\cos(\ln(t))}{t\sin(t)} \end{aligned}$$

(e) $g(x) = \displaystyle\int_0^x \cos(\sqrt{t})\, dt$

$$g'(x) = \cos(\sqrt{x})$$

6. [5 marks] (a) Differentiate implicitly to find the derivative of y (with respect to x) if

$$\frac{1}{2x} + \sqrt{y+2} = 1.$$

$$\begin{aligned} \frac{-1}{2}x^{-2} + \frac{1}{2}(y+2)^{-1/2}y' &= 0 \\ \frac{1}{2}(y+2)^{-1/2}y' &= \frac{1}{2}x^{-2} \\ y' &= \frac{x^{-2}}{(y+2)^{-1/2}} \\ &= \frac{\sqrt{y+2}}{x^2} \end{aligned}$$

(b) Find the equation of the tangent line of the function in part (a) at the point $(\frac{-1}{2}, 2)$.

The slope of the tangent line at the given point:

$$m = y' = \frac{\sqrt{2+2}}{\left(\frac{-1}{2}\right)^2} = \frac{2}{\frac{1}{4}} = 8$$

Then the equation of the tangent line is

$$y - 2 = 8\left(x - \left(\frac{-1}{2}\right)\right) \Longrightarrow y = 8x + 4 + 2 \Longrightarrow y = 8x + 6$$

7. [8 marks] Differentiate with respect to x (use logarithmic differentiation):

(a) $y = \dfrac{(x+1)^{5/2}(x-2)^{1/2}x^3}{e^x}$

$$\begin{aligned} \ln(y) &= \ln\left[\frac{(x+1)^{5/2}(x-2)^{1/2}x^3}{e^x}\right] \\ &= \frac{5}{2}\ln(x+1) + \frac{1}{2}(x-2) + 3\ln(x) - x \end{aligned}$$

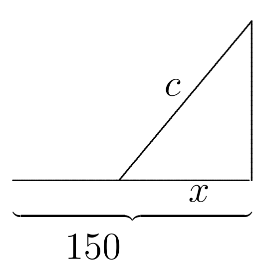
$$\begin{aligned} \frac{1}{y}y' &= \frac{5}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-2} + 3\frac{1}{x} - 1 \\ y' &= y\left[\frac{5}{2(x+1)} + \frac{1}{2(x-2)} + \frac{3}{x} - 1\right] \\ y' &= \left(\frac{(x+1)^{5/2}(x-2)^{1/2}x^3}{e^x}\right)\left[\frac{5}{2(x+1)} + \frac{1}{2(x-2)} + \frac{3}{x} - 1\right] \end{aligned}$$

(b) $y = (\sin(x))^{\ln(x)}$

$$\ln(y) = \ln(x)\ln(\sin(x))$$

$$\begin{aligned} \frac{1}{y}y' &= \frac{1}{x}\ln(\sin(x)) + \ln(x)\frac{1}{\sin(x)}\cos(x) \\ y' &= (\sin(x))^{\ln(x)}\left[\frac{\ln(\sin(x))}{x} + \ln(x)\cot(x)\right] \end{aligned}$$

8. [6 marks] At noon, Ship A is 150 km west of Ship B. Ship A is travelling east at 35 km/hr and Ship B is travelling north at 25 km/hr. How fast is the distance between the ships changing at 4:00pm?



Let y be the distance traveled by Ship B and let $150 - x$ be the distance Ship A has traveled. Then $\frac{dy}{dt} = 25$ km/hr and $\frac{dx}{dt} = -35$ km/hr. Let c be the distance between the two ships. Then

$$c^2 = x^2 + y^2$$

Differentiating

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dc}{dt} = \frac{1}{c}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

At 4:00pm, four hours have passed. So Ship B has traveled $25 \cdot 4 = 100$ km and Ship A has traveled $35 \cdot 4 = 140$. Thus, at 4:00pm, $y = 100$ and $x = 150 - 140 = 10$ and

$$c = \sqrt{x^2 + y^2} = \sqrt{10^{+}100^2} = \sqrt{10100}$$

Therefore,

$$\frac{dc}{dt} = \frac{1}{\sqrt{10100}}\left((10)(-35) + (100)(25)\right) = \frac{2150}{\sqrt{10100}} = \frac{215}{\sqrt{101}}$$

9. [4 marks] Find the linearization $L(x)$ of $f(x) = e^{2x} + x$ for small x, that is, for $a = 0$.

$$f(0) = e^0 + 0 = 1 + 0 = 1$$

$$f'(x) = 2e^{2x} + 1 \Longrightarrow f'(0) = 2e^0 + 1 = 2 + 1 = 3$$

Then

$$L(x) = 1 + 3(x - 0) = 3x + 1$$

10. [4 marks] Find the following limits:

(a)

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

(b) $\lim_{x\to\infty} (e^x + x)^{1/x}$

Let $y = (e^x + x)^{1/x}$. Then $\ln(y) = \frac{1}{x}\ln(e^x + x)$. Therefore,

$$\begin{aligned}
\lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \\
&= \lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x + 1)}{1} \\
&= \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \\
&= \lim_{x\to\infty} \frac{e^x}{e^x + 1} \\
&= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{e^x}} = 1
\end{aligned}$$

Thus, we obtain

$$\lim_{x\to\infty} e^{\ln(y)} = \lim_{x\to\infty} (e^x + x)^{1/x} = e^1$$

11. [8 marks] Given the function $f(x) = x^3 + 6x^2 + 9x$, answer the following:

(a) What is $f'(x)$?

$$f'(x) = 3x^2 + 12x + 9$$

(b) Find the critical numbers of f.

$$f' = 0 \Longrightarrow 3x^2 + 12x + 9 = 0 \Longrightarrow 3(x + 1)(x + 3) = 0$$

Our critical numbers are $x = -1$ and $x = -3$.

(c) Find the intervals of increasing/decreasing.

	$x < -3$	$-3 < x < -1$ ¡ $x > -1$	
f'	$+$	$-$	$+$

f is increasing on $(-\infty, -3)$ and $(-1, \infty)$.

f is decreasing on $(-3, -1)$.

(d) Find the local maximum and/or the local minimum value.

f has a local maximum value at $x = -3$, $f(-3) = 0$. f has a local minimum value at $x = -1$, $f(-1) = -4$

12. [11 marks] Given the function $f(x) = \frac{x}{x-1}$, answer the following:

(a) What is the domain?

$$D = (-\infty, 1) \bigcup (1, \infty)$$

(b) What are the intercepts?

Only intercept is $(0, 0)$.

(c) What is the vertical asymptote? What is the horizontal asymptote?

$$\lim_{x\to\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1$$

So $y = 1$ is a H.A.

$$\lim_{x\to-\infty} \frac{x}{x-1} \cdot \frac{1/x}{1/x} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1$$

$$\lim_{x\to1^-} \frac{x}{x-1} = -\infty$$

So $x = 1$ is a V.A.

$$\lim_{x\to1^+} \frac{x}{x-1} = \infty$$

(d) Find the critical numbers.

$$f'(x) = \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

Since f' is never zero, there are no critical numbers.

(e) Find the intervals of increasing/decreasing.

Since $f' < 0$ for all x where $x \neq 1$, f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

(f) Is there any extreme values?

No extreme values

(g) Find the intervals of concavity.

$$f'' = \frac{2}{(x-1)^3}$$

f'' is never equal to zero.

	$x < 1$	$x > 1$
f"	$-$	$+$

So f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(h) Is there any inflection points?

There are no inflection points.

(i) Using parts (a)-(h), sketch the graph of the function.

13. [5 marks] Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

We want to maximize the area. Let x and y be the dimensions of the rectangle. Then

$$P = 2x + 2y = 100$$

and

$$A = xy.$$

$$2y = 100 - 2x \Longrightarrow y = 50 - x$$

Then

$$A = xy = x(50 - x) = 50x - x^2$$

Differentiating

$$A' = 50 - 2x$$

$$A' = 0 \Longrightarrow 50 - 2x = 0 \Longrightarrow x = \frac{50}{2} = 25$$

Since $A'' = -2 < 0$, by the Second Derivative test, A has a maximum at $x = 25$.

The dimensions are $x = 25$ and $y = 25$.

14. [7 marks] (a) Find the area under the curve of $f(x) = 9 - x^2$ on the interval $[0, 3]$ using 3 approximating rectangles and right endpoints.

$$\Delta x = \frac{3 - 0}{3} = 1$$

$$x_i = 0 + i(1) = i \Longrightarrow x_i = \{1, 2, 3\}$$

$$R_4 = f(1) + f(2) + f(3) = 9 - (1)^2 + 9 - (2)^2 + 9 - (3)^2 = 13$$

(b) State the definition of the definite integral of f from a to b (as the limit of a Riemann sum).

The **definite integral** of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* are the sample points.

(c) Write $\int_0^3 (9 - x^2)\, dx$ as the limit of the Riemann sum over n intervals (Do not evaluate the limit).

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$

Take the sample points to be the right endpoints: $x_i = a + i\Delta x = i\frac{3}{n} = \frac{3i}{n}$. Then from the defn. above.

$$\int_0^3 (9-x^2)\,dx = \lim_{n\to\infty} \sum_{i=2}^{n} \left(9 - \left(\frac{3i}{n}\right)^2\right)\left(\frac{3}{n}\right)$$

15. [10 marks] Evaluate the following integrals.

(a)

$$\begin{aligned}\int_0^{\pi} (\sin(x)+x^2)\,dx &= \left[-\cos(x)+\frac{x^3}{3}\right]_0^{\pi} \\ &= -\cos(\pi)+\frac{\pi^3}{3}-(-\cos(0)+0) \\ &= 2+\frac{\pi^3}{3}\end{aligned}$$

(b) $\int \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta)+C$

(c) $\int 2x\sin(x^2)\,dx$

Let $u = x^2$. Then $du = 2xdx$.

$$\int 2x\sin(x^2)\,dx = \int \sin(u)du = -\cos(u)+C = -\cos(x^2)+C$$

(d) $\int_0^2 x^2\sqrt{1+x^3}\,dx$

Let $u = 1+x^3$. Then $du = 3x^2\,dx \Longrightarrow x^2dx = \frac{1}{3}du$

$$\begin{aligned}x = 0 &\Longrightarrow u = 1+(0)^3 = 1 \\ x = 2 &\Longrightarrow u = 1+(2)^3 = 9\end{aligned}$$

$$\int_0^2 x^2\sqrt{1+x^3}\,dx = \int_1^9 \sqrt{u}\frac{1}{3}du = \frac{2}{9}u^{3/2}\bigg|_1^9 = \frac{2}{9}(9^{3/2}-1) = \frac{52}{9}$$

16. [4 marks] Given that a particle moves in a straight line with velocity $v(t) = 3t^2+4t-6$ with initial displacement $s(0) = 1$, do the following:

(a) by differentiating, find the acceleration function of this particle $a(t)$.

$$a(t) = v'(t) = 6t+4$$

(b) by integrating, find the position function of this particle $s(t)$.

$$s(t) = \int v(t)\,dt = t^3 + 2t^2 - 6t + C$$

$$s(0) = 1 \Longrightarrow C = 1$$

Thus,

$$s(t) = t^3 + 2t^2 - 6t + 1$$

(c) using part part (b), find the position of the particle when $t = 2$.

$$s(2) = 5$$

Total Marks: 100

(BONUS!) 17. (a) Find the derivative of

$$g(x) = \int_0^{x^2} \sqrt{1 + r^3}\,dr.$$

$$g'(x) = 2x\sqrt{1 + x^6}$$

(b) Find f given that

$$f''(\theta) = \sin(\theta) + \cos(\theta)$$

and $f(0) = 3$ and $f'(0) = 4$.

$$f' = -\cos(\theta) + \sin(\theta) + C$$

$$f'(0) = 4 \Longrightarrow C = 5$$

$$f = -\sin(\theta) - \cos(\theta) + 5\theta + D$$

$$f(0) = 3 \Longrightarrow D = 1$$

$$f = -\sin(\theta) - \cos(\theta) + 5\theta + 1$$