Math 1000 First Midterm Exam Solutions
Wednesday, July 22, 2009

1. Consider $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x < 2 \\ x-1 & \text{if } 2 \le x < 3 \\ \frac{1}{x} & \text{if } x \ge 3 \end{cases}$.

   (6 marks) (a)Find each of the following limits:

   (i) $\lim_{x\to 2^-} f(x) = 3$

   (ii) $\lim_{x\to 2^+} f(x) = 1$

   (iii) $\lim_{x\to 2} f(x)$ does not exist

   (iv) $\lim_{x\to 3^-} f(x) = 2$

   (v) $\lim_{x\to 3^+} f(x) = \frac{1}{3}$

   (vi) $\lim_{x\to 3} f(x)$ does not exist

   (1 mark) (b) Give the intervals over which $f(x)$ is continuous.

   Solution. $f(x)$ is continuous over $(-\infty, 2)$, $(2, 3)$ and $(3, \infty)$.

2. Evaluate the following limits:

   (3 marks) (a)
$$\lim_{t\to -3} \frac{2t^2+7t+3}{t^2-9}$$

   Solution.
$$\frac{2t^2+7t+3}{t^2-9} = \frac{(2t+1)(t+3)}{(t-3)(t+3)} = \frac{2t+1}{t-3}, \text{ when } t \neq -3.$$

   Thus,
$$\lim_{t\to -3} \frac{2t^2+7t+3}{t^2-9} = \lim_{t\to -3} \frac{2t+1}{t-3} = \frac{5}{6}.$$

   (3 marks) (b)
$$\lim_{x\to 3} \frac{\sqrt{x^2+16}-4}{x^2}$$

   Solution. The limit of the denominator is not 0, so
$$\lim_{x\to 3} \frac{\sqrt{x^2+16}-4}{x^2} = \frac{\sqrt{25}-4}{3^2} = \frac{1}{9}.$$

   (3 marks) (c)
$$\lim_{\theta\to 0} \frac{\theta^2}{\sin(\theta)\cos(\theta)}$$

Solution.

$$\lim_{\theta\to 0}\frac{\theta^2}{\sin(\theta)\cos(\theta)} = \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)}\cdot\lim_{\theta\to 0}\frac{\theta}{\cos(\theta)} = 1\cdot 0 = 0$$

3. (2 mark) (a) Give the limit definition of the derivative of a function f at a number a.

   Solution.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ or } f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

   (3 marks) (b) Find the equation of the tangent line to the curve $y = (x+3)^2$ at the point $(-1, 4)$ **using the limit definition of derivative** from above.

   Solution. Using the latter definition from part (a) we get

$$f'(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{(2+h)^2-4}{h} = \lim_{h\to 0}\frac{4h+h^2}{h} = \lim_{h\to 0} 4+h = 4,$$

   which is the slope of the requested tangent line. The equation of the tangent line is thus $y-(4) = 4(x-(-1)) \Rightarrow y = 4x+8$.

(4 marks) 4. Use the Intermediate Value Theorem to show that there is a root of

$$f(x) = x^3 - 7x + 1,$$

   that is find a solution to $x^3 - 7x + 1 = 0$, in the interval $(0, 2)$.

   Solution. First note that $f(x)$ is a polynomial and is thus continuous on the given interval. Also, $f(0) = 1 > 0 > -5 = f(2)$ and so the conditions of the IVT are satisfied. Thus, we conclude that there is a root of $f(x)$ in the interval $(0, 2)$.

5. Differentiate the following functions with respect to x:

   (3 marks) (a) $H(x) = x^8 + 3x^2 + \sin(ax) + \sqrt{5}$, where a is a constant.

   Solution. By power, power, chain and constant rules,

$$H'(x) = 8x^7 + 6x + \cos(ax)\cdot a + 0 = 8x^7 + 6x + a\cos(ax).$$

   (3 marks) (b) $f(x) = e^{3x}\sqrt{x}$

   Solution. By the product rule,

$$f'(x) = e^{3x}\cdot 3\cdot\sqrt{x} + e^{3x}\cdot\frac{1}{2\sqrt{x}} = e^{3x}(3\sqrt{x} + \frac{1}{2\sqrt{x}}).$$

   (3 marks) (c) $y = \frac{\ln(x)}{e^{x^2}}$

   Solution. Using the quotient rule,

$$y' = \frac{\frac{1}{x}e^{x^2} - e^{x^2}\cdot 2x\cdot\ln x}{(e^{x^2})^2} = \frac{\frac{1}{x} - 2x\ln x}{e^{x^2}}.$$

(3 marks) (d) $f(x) = \arcsin(t^2)$

Solution. With respect to x, $\arcsin(t^2)$ is a constant thus $f'(x) = 0$.

(3 marks) (d*) $f(x) = \arcsin(x^2)$

Solution. Using the chain rule,

$$f'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}.$$

(3 marks) (e) $G(x) = \ln(\sin(\frac{1}{x}))$

Solution. Using the chain rule,

$$G'(x) = \frac{1}{\sin(\frac{1}{x})} \cdot \cos(\frac{1}{x}) \cdot \frac{-1}{x^2} = \frac{-\cot(\frac{1}{x})}{x^2}.$$

(3 marks) 6. Find the equation of the tangent line to the curve $y = \tan(x)$ at the point $(\frac{\pi}{3}, \sqrt{3})$.

Solution. To find the equation of the line we want the slope so we take the derivative. We find that $y' = \sec^2(x)$. Evaluating y' at $\frac{\pi}{3}$ gives $m = 4$. Thus the equation of our line is

$$y - \sqrt{3} = 4(x - \frac{\pi}{3}) \Rightarrow y = 4x - \frac{4\pi}{3} + \sqrt{3}.$$

(4 marks) 7. Find y'' if $x^6 + y^6 = 1$.

Solution. Taking the derivative of both sides implicitly we get $6x^5 + 6y^5y' = 0$. Solving for y' we find

$$y' = -\frac{x^5}{y^5} = -\left(\frac{x}{y}\right)^5.$$

We now take the derivative of both sides of this equation, using the chain rule on the right, to find y''. We get

$$y'' = -5\left(\frac{x}{y}\right)^4 \cdot \left(\frac{1 \cdot y + x \cdot y'}{y^2}\right) = -5\left(\frac{x^4}{y^5} + \frac{x^{10}}{y^{11}}\right).$$

8. Use logarithmic differentiation to find the derivative of the function:

(4 marks) (a) $y = (\sin(x))^x$

Solution. Take the natural log of both sides to get

$$\ln y = \ln\left([\sin(x)]^x\right) = x\ln(\sin(x)).$$

Taking the derivative of each side gives

$$\frac{1}{y}y' = 1 \cdot \ln(\sin(x)) + x\frac{1}{\sin(x)}\cos(x)$$

which gives

$$y' = (\sin(x))^x \left(\ln(\sin(x)) + x\cot(x)\right).$$

(4 marks) (b) $y = \frac{(x^2+1)^4}{(2x+1)^3 e^{5x}}$

Solution. Take the natural log of both sides to get

$$\ln y = \ln\left(\frac{(x^2+1)^4}{(2x+1)^3 e^{5x}}\right) = 4\ln(x^2+1) - 3\ln(2x+1) - 5x.$$

Taking the derivative of each side gives

$$\frac{1}{y}y' = 4\frac{1}{x^2+1}2x - 3\frac{1}{2x+1}2 - 5$$

which gives

$$y' = \frac{(x^2+1)^4}{(2x+1)^3 e^{5x}}\left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - 5\right).$$

(4 marks) 9. The position of a particle at time t is given by $s(t) = 5t^2 - 2t + 3$. Find the velocity of the particle at time $t = 3$.

Solution. Velocity is

$$v(t) = s'(t) = 10t - 2.$$

Subbing in $t = 3$ gives the velocity at time 3 which is

$$v(3) = 30 - 2 = 28.$$

(5 marks) 10. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm? (Note: the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.)

Solution. From the question we are given

$$\frac{dr}{dt} = 4.$$

We are asked to find

$$\frac{dV}{dt} \text{ when } r = 40.$$

Taking the derivative of both sides of the given equation for volume with respect to time, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}.$$

Subbing in r and $\frac{dr}{dt}$ we find

$$\frac{dV}{dt} = 4\pi(40)^2 4 = 25600\pi\frac{\text{mm}^3}{\text{s}} \text{ when the diameter is 40 mm.}$$

Total Marks: 64 + 3 bonus marks.