Math 1000 Second Midterm Exam Solutions

Wednesday, August 5, 2009

1. (a) State the formula for $L(x)$, the linearization of the function $f(x)$ at a. [2]

   Solution. $L(x) = f(a) + f'(a)(x - a)$ is the linearization of the function $f(x)$ at a.

   (b) Find the linear approximation of the function $f(x) = \sin(x)$ at 0. [2]

   Solution. $\sin(x) \approx \sin(0) + \cos(0)(x - 0) = 0 + 1 \cdot x = x$ at 0.

   (c) Given that $e^{x-1} \approx x$ for values of x near 1, give an approximation of $e^{0.001}$. [2]

   Solution. Given that $e^{x-1} \approx x$, and that $e^{0.001} = e^{x-1}$ when $x = 0.001$, then $e^{0.001} \approx 1.001$.

2. Find the critical numbers of [5]

$$f(x) = \sqrt{x} - \frac{1}{2}x.$$

   Solution. The critical numbers of f are the values of c in the domain of f such that $f'(c) = 0$ or such that $f'(c)$ is undefined. We find the derivative to be $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}$. Let $f'(x) = 0$, that is $0 = \frac{1}{2\sqrt{x}} - \frac{1}{2}$ which implies $\sqrt{x} = 1$ and thus $x = 1$. Now we consider where the derivative is not defined. The domain is of f $x \geq 0$, but the derivative is only defined for $x > 0$, so 0 is also a critical number. That is 0 and 1 are the critical numbers of f.

3. Apply the Closed Interval Method to find the absolute maximum and minimum values of $f(x) = 3x^2 - 12x + 5$ on the interval $[0, 3]$. [5]

   Solution. First we find that the derivative is $f'(x) = 6x - 12$. If we let the derivative be 0 then we find $x = 2$. Thus the closed interval method requires we compare the values of $f(0)$, $f(2)$ and $f(3)$. We find that $f(0) = 5$ is maximum value of f and that $f(2) = -7$ is the minimum value. For completeness we note $f(3) = -4$.

4. Verify that the function $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem. [5]

   Solution. The function f is a polynomial and thus is continuous and differentiable everywhere, thus it satisfies the hypotheses of the MVT. We find $f(b) = f(2) = 9$ and $f(a) = f(0) = -1$. Thus we are looking for c such that

$$f'(c) = \frac{9 - (-1)}{2 - 0} = 5.$$

   That is, we want to solve $3x^2 + 1 = 5$ for x, the solutions of which are $x = \pm\frac{2}{\sqrt{3}}$. However, only $x = \frac{2}{\sqrt{3}}$ is in the given interval.

5. Consider $f(x) = \sin x + \cos x$, for $0 \leq x \leq 2\pi$.

(a)+(b) Find the intervals on which f is increasing or decreasing. [2]

Solution. We find $f'(x) = \cos x - \sin x$. We let $0 = \cos x - \sin x$ and find that $x = \frac{\pi}{4}, \frac{5\pi}{4}$. Moreover, we find that f is increasing on $[0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, 2\pi]$ and decreasing on $(\frac{\pi}{4}, \frac{5\pi}{4})$. By the first derivative test, a local maximum occurs at $\frac{\pi}{4}$ and a local minimum occurs at $\frac{5\pi}{4}$.

(c) Find the intervals of concavity and the inflection points. [2]

Solution. The second derivative is $f''(x) = -\sin x - \cos x$. The second derivative is 0 when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. The function is concave down on $[0, \frac{3\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi]$ and concave up on $(\frac{3\pi}{4}, \frac{7\pi}{4})$. Both $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ are inflection points.

6. Calculate [5]

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}.$$

Solution.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} (\text{type } \frac{\infty}{\infty}) \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3x^{\frac{2}{3}}}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0.$$

7. Calculate [5]

$$\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{(2x+1)}.$$

Solution. This limit is of the form 1^∞, which we will evaluate using ln.

$$\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{(2x+1)} = e^{\lim_{x\to\infty}(2x+1)(\ln(2x-3)-\ln(2x+5))}.$$

The limit in the exponent is of the form $\infty \cdot 0$, which by rewriting and using L'Hospitâl's Rule we get

$$\begin{aligned}
\lim_{x\to\infty}(2x+1)(\ln(2x-3)-\ln(2x+5)) &= \lim_{x\to\infty} \frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}} \\
&\overset{L}{=} \lim_{x\to\infty} \frac{\frac{2}{2x-3} - \frac{2}{2x+5}}{\frac{-1\cdot 2}{(2x+1)^2}} \\
&= \lim_{x\to\infty} \frac{-[(2x+5)-(2x-3)](2x+1)^2}{(2x+5)(2x-3)} \\
&= \lim_{x\to\infty} \frac{-8(2+\frac{1}{x})^2}{(2+\frac{5}{x})(2-\frac{3}{x})} \\
&= \frac{-8\cdot 2\cdot 2}{2\cdot 2} \\
&= -8.
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{(2x+1)} = e^{-8}.$$

8. Verify, given $f(x) = e^{-x}(x^2 - x)$, that [2]

$$f'(x) = -e^{-x}(x^2 - 3x + 1),$$

and that [2]

$$f''(x) = e^{-x}(x^2 - 5x + 4).$$

Solution.

$$f'(x) = -e^{-x}(x^2 - x) + e^{-x}(2x - 1) = -e^{-x}(x^2 - 3x + 1),$$

and

$$f''(x) = e^{-x}(x^2 - 3x + 1) - e^{-x}(2x - 3) = e^{-x}(x^2 - 5x + 4).$$

9. Consider the curve given by the function $f(x) = e^{-x}(x^2 - x)$.

(a) What is the domain of the function f? [2]
Solution. The functions e^{-x} and $x^2 - x$ both have domain the entire real line, so their product also has the domain $\mathbb{R}$.

(b) Does the function have any x-intercepts or y-intercepts? If so, what are they? [2]
Solution. Solving $0 = e^{-x}(x^2 - x)$ gives $x = 0,\ 1$ as x-intercepts. The y-intercept is $f(0) = 0$.

(c) Is the function even? Is the function odd? [2]
Solution. The function is neither even nor odd.

(d) Does the function have any vertical asymptotes? If so, what are they? Does the function have any horizontal asymptotes? If so, what are they? [2]
Solution. The function has no vertical asymptotes as it is continuous on $\mathbb{R}$. As for horizontal asymptotes,

$$\lim_{x\to\infty} e^{-x}(x^2 - x) = 0,$$

and

$$\lim_{x\to-\infty} e^{-x}(x^2 - x) = \infty,$$

so there is one horizontal asymptote, $y = 0$.

(e) Find the critical numbers of f. Where is the function increasing? Where is the function decreasing? [2]
Solution. We have $f'(x) = -e^{-x}(x^2 - 3x + 1)$. The critical numbers of f are the values of x for which $x^2 - 3x + 1 = 0$, which are $x = \frac{3\pm\sqrt{5}}{2}$. We find the function is decreasing on $(-\infty, \frac{3-\sqrt{5}}{2}) \cup (\frac{3+\sqrt{5}}{2}, \infty)$ and increasing on $(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$.

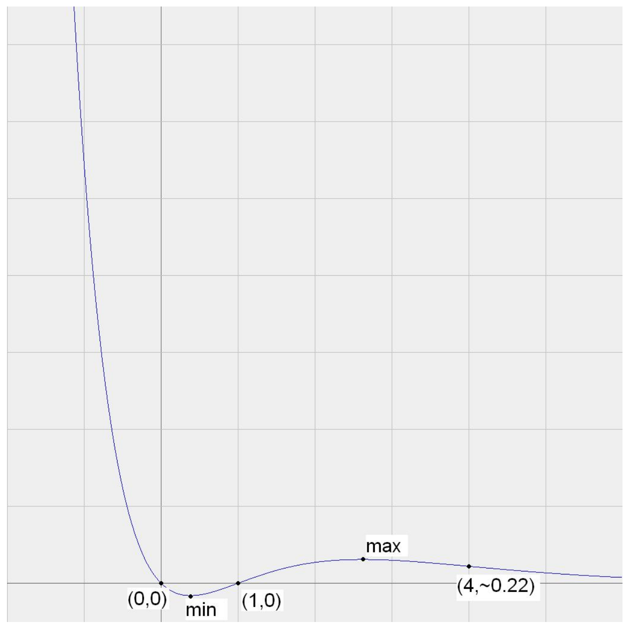
(f) Find all local maximum and minimum values of f. Use either the first or second derivative test to justify your answer. [2]
Solution. By the first derivative test, we see from the previous part that there is a local minimum at $\frac{3-\sqrt{5}}{2}$ and that there is a local maximum at $\frac{3+\sqrt{5}}{2}$.

(g) Determine where the curve is concave up and where the curve is concave down. Does the curve have any inflection points? If so, what are they? [2]

Solution. The second derivative is $f''(x) = e^{-x}(x^2 - 5x + 4)$, which takes the value 0 when $x = 1,\ 4$. The function f is concave up on $(\infty, 1) \cup (4, \infty)$ and concave down on $(1, 4)$. There are inflection points at both $x = 1$ and $x = 4$.

(h) Sketch the graph of the function. Plot all intercepts, asymptotes, local maxima and minima, and inflection points. [2]

Solution.



Total: $\overline{57}$