Chapter 5

Integrals

5.1 Areas and Distances

The Area Problem

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

[picture]

We know how to find the areas of regions with straight sides such as triangles, rectangles and other polygons. How do we find the area of other regions? We take what we already know and use it as best we can: we approximate areas by polygons, usually many rectangles.

Example 156. Consider the area under the curve $y = x^2$.

[lots of space]

Problem 157. Estimate the area under the graph of $f(x) = 2 - x^2$ from $x = 0$ to $x = 1$ using four approximating rectangles and right endpoints and then left endpoints.

Solution.

[lots of space]

In both of these examples it is fairly easy to see that more and more rectangles would give better and better approximations. See the text for nice pictures.

In the tangent problem, which lead to our definition of derivative, we found ourselves looking at secant line approximations that were based on points that were closer and closer to where our line was to be tangent. In the area problem, we find ourselves looking at approximations based on increasing numbers of rectangles. Again, we will get a definition based on limits. We formalize this below.

For the area under the graph of a general function $y = f(x)$ on the interval $[a, b]$ with n approximating rectangles the width of each rectangle is

$$\Delta x = \frac{\text{width of the interval}}{\text{number of rectangles}} = \frac{b-a}{n}.$$

These n rectangles divide the interval $[a, b]$ into n subintervals

$$[a = x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-2}, x_{n-1}],\ [x_{n-1}, x_n = b].$$

The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \\ x_i &= a + i\Delta x \\ &\vdots \\ x_n &= a + n\Delta x \end{aligned}$$

So $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$.

As we take more and more rectangles, the estimates from the right endpoints and the estimates from the left endpoints come closer together.

From this, we get the following definition.

Definition 158. The *area* A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles. That is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x].$$

We get the same value if we use left endpoints.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x].$$

Instead of using right endpoints or left endpoints, we could take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval, and again we would get the same value. We call the numbers $x_1^*,\ x_2^*, \ldots,\ x_n^*$ the sample points.

So a more general expression is

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x].$$

Before we do an example, we introduce some notation to make things easier to write. **Sigma notation** is used to write sums with many terms (as above) more compactly. Appendix E in the text has more. For example,

$$\sum_{i=1}^{n} f(x_i) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

Also, from our first example, we would have

$$R_4 = \sum_{i=1}^{4} \frac{1}{4} f(\frac{1}{4} i).$$

$$\sum_{i=m}^{n} f(x_i)\Delta x$$

Then our general area limit becomes:

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Problem 159 (Exercise 17, page 365)**.** Use the definition to find an expression, using limits, for the area under the graph of $f(x) = \sqrt[4]{x}$ on the interval $[1, 16]$.

Solution.

Example 160. Determine a region whose area is equal to the given limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan(\frac{i\pi}{4n})$$

Solution.

5.2 The Definite Integral

In the previous section, we saw that a limit of the form

$$\lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

arises when we compute an area.

It turns out that this type of limit occurs in a wide number of situation. As such, we give it a special name and notation.

Definition 161. If f is a function defined for $a \leq x]eb$, we dived the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $a = x_0,\ x_1,\ x_2, \ldots, x_n = b$ be the endpoints of these subintervals, and we let $x_1^*,\ x_2^*, \ldots, x_n^*$ be any *sample points* in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the *definite integral of f from a to b* is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. If it does exist, we say that f is *integrable* on $[a, b]$.

Notation. In

$$\int_a^b f(x)dx$$

, we call $\int$ the *integral sign*, $f(x)$ the *integrand*, a and b the *bounds (or limits) or integration*, and dx represents the variable with which we are integrating. Also, we call $\sum_{i=1}^{n} f(x_i^*)\Delta x$ the *Riemann sum*.

Note. The definite integral is a number; so

$$\int_a^b f(x)dx = \int_a^b f(r)dr = \int_a^b f(t)dt.$$

We know if f is positive, the Riemann sum can be interpreted as a sum of areas of approximating rectangles.

If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$ is the area under the curve $y = f(x)$ from a to b.

If f takes both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles below the x-axis. A definite integral can be interpreted as a *net area*, that is, a difference of areas:

$$\int_a^b f(x)dx = A_z - A_2,$$

where a_1 is the area above the x-axis and below the graph of y, and A_2 is the area below the x-axis and above the graph of y.

Not all functions are integrable, but most of the functions we look at in this course are, at least for most intervals in their domain.

Theorem 162. *If f is continuous on $[a, b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, $\int_a^b f(x)dx$ exists.*

Theorem 163 (Theorem 4)**.** *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Problem 164 (Exercise 19, page 377)**.** Express the limit as a definite integral on the given subinterval.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2x_i^8 + (x_i^*)^2}\Delta x, \ [1, 8].$$

Solution.

Problem 165 (Exercise 29, page 377)**.** Express

$$\int_2^6 \frac{x}{1+x^5}dx$$

as the limit of Riemann sums.

Solution.

How do we evaluate these limits? Note that if we already had the tools to do so we would already be doing it. We need to know the following first.

We have formulas to rewrite these sums into something more manageable.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Then some rules for working with sigma notation:

$$\sum_{i=1}^{n} c = n \cdot c$$
$$\sum_{i=1}^{n} c \cdot a_i = c\sum_{i=1}^{n} a_i$$
$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Problem 166. Use Theorem 4 to evaluate the integral

$$\int_0^2 (1+4x^3)dx.$$

Solution.

Theorem 167 (Midpoint Rule)**.**

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

Problem 168 (Exercise 12, page 377)**.** Use the Midpoint Rule with the $n = 4$ to approximate

$$\int_1^5 x^2 e^{-x} dx.$$

Solution.

Some quick properties of the definite integral:

$$\int_b^a f(x)dx = \int_a^b -f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b cdx = c(b-a), \text{ where } c \text{ is any constant.}$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b c[f(x)]dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) links the two branches of calculus: differential calculus (this course mostly) and integral calculus (this chapter and Math 1010). It does so by describing how integration