Definition 34. The *velocity* (or instantaneous velocity) $v(a)$ at time $t = a$ is the limit of the average velocities over shorter and shorter time intervals $[a, a + h]$:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition 35. The *derivative of a function* f *at a number* a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

An equivalent definition is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Rates of Change

If y is a quantity that depends on another quantity x, then y is a function of x ($y = f(x)$). If x changes from x_1 to x_2, then the change of x is $\Delta x = x_2 - x_1$, and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$.

Then the *average rate of change of* y *with respect to* x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now, then instantaneous rate of change of y with respect to x is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Problem 36 (Exercise 19, page 151)**.** Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

Solution.

Problem 37 (Exercise 29, page 151)**.** Find $f'(a)$ where

$$f(x) = \frac{1}{\sqrt{x+2}}.$$

Solution.

2.8 The Derivative as a Function

We have considered the derivative of a function f at a fixed point a. Now we let a vary. For any number x, let

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We consider $f'(x)$ a new function, which we call the *derivative of* f. We can interpret $f'(x)$ geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Problem 38 (Example 4, page 156)**.** Find f' if

$$f(x) = \frac{1-x}{2+x}.$$

Solution.

Other notations of differentiation: Given $y = f(x)$, then $f' = f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$.

Definition 39. A function f is *differentiable at a* if $f'(a)$ exists. It is *differentiable on an open interval* (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Note. An open interval (a, b) is the set of real numbers strictly greater than a and less than b.

Theorem 40. *If f is differentiable at a, then f is continuous at a.*

There are, however, functions that are continuous but not differentiable.

Example 41. Consider the function $f(x) = |x|$. The function is clearly continuous on $(-\infty, 0)$ and $(0, \infty)$ and it can be shown easily that it is also continuous at 0. The function is also differentiable on $(-\infty, 0)$ and $(0, \infty)$, but it not differentiable at 0. This is understood by considering the many lines that can be tangent to the curve at 0, all with different slopes. That is, there is no unique slope of the tangent line to the curve at that point and so the derivative is undefined. [picture]

Higher Derivatives

Definition 42. The *second derivative* of a function f is the derivative of the derivative, which we denote as f'', $\frac{d^2f}{dx^2}$.

We can think of the second derivative as:

- the slope of the curve $f'(x)$ [i.e. the rate of change of the slope of the derivative]
- acceleration: if $s = s(t)$ is the position function of an object that moves in a straight line, the velocity is

$$v(t) = s'(t)$$

  and the acceleration is

$$a(t) = v'(t) = s''(t).$$

The *third derivative* of f is the derivative of the second derivative, which we denote $f'''(x)$; we also get the formula $f'''(x) = (f''(x))'$.

This process can continue indefinitely, to get higher and higher derivatives. We denote the nth derivative as

$$y^{(n)} = f^{(n)} = \frac{d^n y}{dx^n}.$$

Example 43 (Example 7, page 161)**.** Consider $f(x) = x^3 - x$.

Solution.

Chapter 3

Differentiation Rules

Using the limit definition of the derivative can be tedious. We will now learn rules to determine the derivative of functions easily.

3.1 Derivatives of Polynomials and Exponential Functions

Theorem 44 (Derivative of a constant function)**.** *The derivative of a constant function is zero, that is*

$$\frac{d}{dx}(c) = 0.$$

Proof. Let $f(x) = c$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(c) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Theorem 45 (The Power Rule)**.**

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof.

3.1.1 Examples

1. If $f(x) = x^{100}$, then $f'(x) = 100x^{99}$.
2. If $f(x) = \sqrt{x}$, then
$$f'(x) = (\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$
3. If $f(x) = \frac{1}{x^2}$, then
$$f'(x) = (x^{-2})' = -2x^{-3} = \frac{-2}{x^3}.$$

Problem 46. Find equations of the tangent line and the normal line to the curve $y = \sqrt[4]{x^3}$ at $(1, 1)$.

Solution.

3.2 New Derivatives From Old

Theorem 47 (The Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Theorem 48 (The Sum Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Theorem 49 (The Difference Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Problem 50 (Exercise 49, page 181)**.** The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds.

- Find the velocity and acceleration as functions of t.
- Find the acceleration after $2s$.
- Find the acceleration when the velocity is 0.

Solution.

3.3 Exponential Functions

Definition 51 (The number e)**.** Let e be the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Theorem 52 (Derivative of the Natural Exponential Function)**.**

$$\frac{d}{dx}e^x = e^x.$$

Problem 53. Find the first and second derivatives of $f(x) = e^x - x$.

Solution.

Problem 54 (Exercise 9, page 181)**.** At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Solution.

Problem 55. Consider the curve $y = 2x^3 + 3x^2 - 12x + 1$. Find all the point on this curve where the tangent is horizontal.

Solution.

3.4 The Product and Quotient Rule

Theorem 56 (The Product Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)].$$

Notation. The Product Rule is also often written as:

$$(fg)' = f'g + g'f.$$

Problem 57. Find $f'(x)$ if $f(x) = x^2e^x$.

Solution.

Problem 58. Find $f'(t)$ if $f(t) = \sqrt{t}(a + bt)$.

Solution.

Theorem 59 (The Quotient Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]g(x) - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Problem 60. Differentiate $g(x) = \frac{3x-1}{2x+1}$.

Solution.

Problem 61. Differentiate $F(x) = \frac{x-3x\sqrt{x}}{\sqrt{x}}$.

Solution.

Problem 62. Find and equation of the tangent line to the curve $y = \frac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

Solution.

3.5 Derivatives of Trigonometric Functions

Fact 63 (Two Special Limits)**.**

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

and

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

Problem 64 (Exercise 39, page 196)**.** Evaluate

$$\lim_{x \to 0} \frac{\sin 3x}{x}.$$

Solution.

Problem 65 (Exercise 41, page 196)**.** Evaluate

$$\lim_{t \to 0} \frac{\tan 6t}{\sin 2t}.$$

Solution.

Problem 66 (Exercise 41, page 196)**.** Evaluate

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta}.$$

Solution.

Using these two limits, we can also prove the formula for the derivative of the sine function.

Theorem 67.

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof.

Theorem 68.

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Theorem 69.

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = \csc^2(x)$$

Figure 3.1: Summary of Derivatives of Trigonometric Functions

Proof.

Similarly, we can also find the derivatives of $\sec(x)$, $\csc(x)$, and $\cot(x)$.

Problem 70 (Exercise 24, page 195)**.** Find the equation of the tangent line to the curve $y = \frac{1}{\sin x+\cos x}$ at $(0, 1)$.

Solution.