Chapter 3

Differentiation Rules

Using the limit definition of the derivative can be tedious. We will now learn rules to determine the derivative of functions easily.

3.1 Derivatives of Polynomials and Exponential Functions

Theorem 44 (Derivative of a constant function)**.** *The derivative of a constant function is zero, that is*

$$\frac{d}{dx}(c) = 0.$$

Proof. Let $f(x) = c$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(c) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Theorem 45 (The Power Rule)**.**

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof.

3.1.1 Examples

1. If $f(x) = x^{100}$, then $f'(x) = 100x^{99}$.
2. If $f(x) = \sqrt{x}$, then
$$f'(x) = (\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$
3. If $f(x) = \frac{1}{x^2}$, then
$$f'(x) = (x^{-2})' = -2x^{-3} = \frac{-2}{x^3}.$$

Problem 46. Find equations of the tangent line and the normal line to the curve $y = \sqrt[4]{x^3}$ at $(1,1)$.

Solution.

3.1.2 New Derivatives From Old

Theorem 47 (The Constant Multiple Rule)**.** *If c is a constant and f is a differentiable function, then*
$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$

Theorem 48 (The Sum Rule)**.** *If f and g are both differentiable, then*
$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Theorem 49 (The Difference Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Problem 50 (Exercise 49, page 181)**.** The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds.

- Find the velocity and acceleration as functions of t.
- Find the acceleration after $2s$.
- Find the acceleration when the velocity is 0.

Solution.

3.1.3 Exponential Functions

Definition 51 (The number e)**.** Let e be the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Theorem 52 (Derivative of the Natural Exponential Function)**.**

$$\frac{d}{dx}e^x = e^x.$$

Problem 53. Find the first and second derivatives of $f(x) = e^x - x$.

Solution.

Problem 54 (Exercise 9, page 181)**.** At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Solution.

Problem 55. Consider the curve $y = 2x^3 + 3x^2 - 12x + 1$. Find all the point on this curve where the tangent is horizontal.

Solution.

3.2 The Product and Quotient Rule

Theorem 56 (The Product Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)].$$

Notation. The Product Rule is also often written as:

$$(fg)' = f'g + g'f.$$

Problem 57. Find $f'(x)$ if $f(x) = x^2e^x$.

Solution.

Problem 58. Find $f'(t)$ if $f(t) = \sqrt{t}(a + bt)$.

Solution.

Theorem 59 (The Quotient Rule)**.** *If f and g are both differentiable, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]g(x) - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Problem 60. Differentiate $g(x) = \frac{3x-1}{2x+1}$.

Solution.

Problem 61. Differentiate $F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$.

Solution.

Problem 62. Find and equation of the tangent line to the curve $y = \frac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

Solution.

3.3 Derivatives of Trigonometric Functions

Fact 63 (Two Special Limits)**.**

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

and

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.$$

Problem 64 (Exercise 39, page 196)**.** Evaluate

$$\lim_{x \to 0} \frac{\sin 3x}{x}.$$

Solution.

Problem 65 (Exercise 41, page 196)**.** Evaluate

$$\lim_{t \to 0} \frac{\tan 6t}{\sin 2t}.$$

Solution.

Problem 66 (Exercise 41, page 196)**.** Evaluate

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta}.$$

Solution.

Using these two limits, we can also prove the formula for the derivative of the sine function.

Theorem 67.

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof.

Theorem 68.

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Theorem 69.

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = \csc^2(x)$$

Figure 3.1: Summary of Derivatives of Trigonometric Functions

Proof.

Similarly, we can also find the derivatives of $\sec(x)$, $\csc(x)$, and $\cot(x)$.

Problem 70 (Exercise 24, page 195)**.** Find the equation of the tangent line to the curve $y = \frac{1}{\sin x + \cos x}$ at $(0, 1)$.

Solution.

Since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$, we obtain a particular pattern with the higher derivatives.

Problem 71 (Example 4, page 194)**.** Find the 27th derivative of $\cos x$.

Solution.

3.4 The Chain Rule

Still, there are many function that we do not know how to differentiate, such as

$$\sqrt{x^2-1} \text{ or } e^{x^2} \text{ or } \sin(t^2+1).$$

We notice that all of these functions are composite functions.

Definition 72. Given two functions f and g, the *composite function* $f \circ g$ (also called the *composition* of f and g) is defined by

$$(f \circ g)(x) = f(g(x)).$$

Example 73. Consider the function

$$F(x) = \sqrt{x^2-1}.$$

We know how to differentiate f and g, so it would be useful to be able to write $F'(x)$ in terms of f' and g'.

Theorem 74 (The Chain Rule)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Problem 75. Find $F'(x)$ if $F(x) = \sqrt{x^2 - 1}$.

Solution.

Problem 76. Differentiate $\cos(x^2)$ and $\cos^2(x)$.

Solution.

Problem 77. Find $f'(x)$ of $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.

Solution.

Problem 78. Differentiate $y = (2x-5)^3(8x^2-5)^{-3}$.

Solution.

3.4.1 Exponential Functions with Different Bases

Recall that $\frac{d}{dx}e^x = e^x$. But what is the derivative of a^x where $a \neq e$.

Theorem 79. *The derivative of the exponential function with base a, is given by*

$$\frac{d}{dx}a^x = a^x \ln a.$$

Proof.

Problem 80. Find $f'(x)$ if $f(x) = \frac{3}{2}^x$.

Solution.

3.4.2 Using Chain Rule more than once

Problem 81. Differentiate

$$y = \cos(\sqrt{\sin(\tan(\pi x))}).$$

Solution.

3.5 Implicit Differentiation

All of the functions we have considered so far can be described by expressing one variable explicitly in terms of another variable such as

$$y = xcosx \text{ or } y = \sqrt{x^3 + 1} \text{ or } y = f(x).$$

Some functions are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25 \text{ or } x^3 + y^3 = 6xy.$$

For these we cannot always solve for y in terms of x. We call the method we use to find derivatives of such functions *implicit differentiation*}.