Problem 83. Find y' if

$$\sin(x + y) = y^2 \cos(x).$$

Solution.

Problem 84. Find y'' if $x^4 + y^4 = 16$.

Solution.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2+1} \qquad \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{x^2+1}$$

Figure 3.2: Summary of Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

We can use implicit differentiation to obtain the formulas for the derivatives of the inverse trig. functions.

Recall that $y = \sin^{-1}(x)($ or $\arcsin(x)$ means $\sin y = x$, $\frac{-\pi}{2} \le y \le \frac{pi}{2}$.

Example 85. Differentiate (implicitly) $\sin y = x$.

Solution.

Problem 86. Find y' if $y = \arcsin x^2$.

Solution.

Problem 87 (Exercise 49, page 214)**.** Differentiate $G(x) = \sqrt{1-x^2}\arccos x$.

Solution.

3.6 Derivatives of Logarithmic Functions

Theorem 88.

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

In particular, if $a = e$, then

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Example 89.

$$\frac{d}{dx}\ln(\sin x) =$$

$$\frac{d}{dx}\sqrt{\ln x} =$$

Problem 90 (Exercise 10, page 220)**.** Differentiate

$$f(t) = \frac{1+\ln t}{1-\ln t}.$$

Solution.

Logarithmic Differentiation

For the most complicated functions, often taking logarithms helps make our calculations much simpler.

We will do an example and then explain the general method.

First, a quick review of laws of logarithms:

Problem 91 (Exercise 38, page 220)**.** Find y' if $y = \sqrt{x}e^{x^2}(x^2+1)^{10}$.

Solution.

$$\begin{aligned}
\ln(ab) &= \ln(a) + \ln(b) \\
\ln(\frac{a}{b}) &= \ln(a) - \ln(b) \\
\ln(a^r) &= r\ln(a) \\
\ln(e) &= 1 \\
e^{\ln x} &= x
\end{aligned}$$

Figure 3.3: Some Logarithm Rules

General Method

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

We now have four different ways to handle different types of derivatives involving exponents:

1. $\frac{d}{dx}a^b = 0$ (a, b are constants)
2. $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$ (use chain rule and the power rule)
3. $\frac{d}{dx}a^{g(x)} = a^{g(x)}\ln(a)g'(x)$ (use chain rule and the rule for exponents with base other than e)
4. $\frac{d}{dx}[f(x)]^{g(x)} \rightarrow$ use logarithmic differentiation

Problem 92 (Exercise 43, page 220)**.** Find $\frac{d}{dx}x^{\sin x}$.

Solution.

Problem 93 (Exercise 45, page 220). Differentiate $y = (\cos x)^x$.

Solution.

3.7 Rates of Change in the Natural Sciences

Whenever the function $y = f(x)$ has a specific interpretation in one of the sciences, its derivative will have a specific interpretation as a rate of change.

Velocity and Acceleration Recall: if $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\frac{\Delta s}{\Delta t}$ represents the average velocity over a time period Δt, and $v = \frac{ds}{dt}$ represents the instantaneous velocity. The instantaneous rate of change of velocity with respect to time is acceleration.

Problem 94 (Exercise 1, page 230). A particle moves according to a law of motion $s = f(t) = t^3 - 12t^2 + 36t$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) Find velocity at time t.

(b) What is the velocity after $3s$?

(c) When is the particle at rest?

(d) When is the particle moving in the positive direction?

(e) Find the total distance traveled in the first $8s$.

(g) Find the acceleration at time t.

(i) When is the particle speeding up? When is it slowing down?

Solution.

Biology The growth rate is the derivative of the population.

Example 95 (Exercise 23, page 232)**.** Consider a bacteria population that triples every hour and starts with 400 bacteria. Find an expression for the number n of bacteria after t hours and use it to estimate the rate of growth after 2.5 hours.

Solution.

3.8 Exponential Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to y is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky,$$

where k is a constant. (This equation is called the law of natural growth.)

It is called a differential equation because it involves an unknown function y and its derivative $\frac{dy}{dt}$.

Theorem 96. *The only solutions of the differential equation* $\frac{dy}{dt} = ky$ *are the exponential functions*

$$y(t) = y(0)e^{kt}.$$

Population Growth

In the context of population growth, where $f(t)$ is the size of a population, we can write

$$\frac{dP}{dt} = kP.$$

Example 97 (Exercise 2, page 239)**.** In a particular bacterium, a cell divides into two cells every 20 minutes. The initial population of a culture is 60 cells. What is the relative growth rate?

Solution.

Newton's Law of Cooling

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings .

If we let $T(t)$ be the temperature of the object at time t and let T_s be the temperature of the surroundings, then Newton's law of cooling is:

$$\frac{dT}{dt} = k(T - T_s).$$

Example 98 (Exercise 13, page 240)**.** A turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after $\frac{1}{2}$ hour, what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 100°F?

Solution.

3.9 Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity which may be more easily measured.

Problem 99 (Exercise 7, page 245)**.** If $y = x^3 + 2x$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$.

Solution.

Problem 100 (Exercise 2, page 245)**.**

(a) If A is the area of a circle with radius r and the circle expands as time passes, find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of $1m/s$, how fast is the area of the spill increasing when the radius is $30m$?

Problem 101 (Example 2, page 242)**.** A ladder 10ft long rests against a vertical wall. If the ladder slides away from the wall at a rate of 1ft/s how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6ft from the wall?

Solution.

Problem 102 (Exercise 16, page 245)**.** A spotlight on the ground shines on a wall 12m away. If a man 2m tall walks from the spotlight toward the building at a speed of 1.6m/s, how fast is the length of his shadow on the building decreasing when he is 4m from the building?

Solution.