Problem 103 (Exercise 30, page 246)**.** In our previous example with the ladder: how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6ft from the wall?

Solution.

3.10 Linear Approximations

This section is about finding approximate values of functions. The main idea is that while it may be very difficult to compute the function f at a, it may be easy to compute the value of a function that is close to f (at a). In particular, we can approximate the function $f(x)$ near a by its tangent at $(a, f(a))$.

What is the slope of the tangent line at $(a, f(a))$

$$f'(a)$$

So what is the equation of this tangent line?

The equation of a line is

$$y - y_1 = m(x - x_1)$$

so the equation of this tangent line is

$$y - f(a) = f'(a)(x - a).$$

Solving for y we get

$$y = f(a) + f'(a)(x - a).$$

Therefore, for x near a, we can approximate the function by the line $y = f(a) + f'(a)(x - a)$. Which is to say

$$f(x) \approx f(a) + f'(a)(x - a).$$

We can such an approximation the *linear approximation* or *tangent line approximation* of f at a.

The linear function whose graph is this tangent line

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Problem 104 (Exercise 1, page 252)**.** Find the linearization $L(x)$ of the function $f(x) = x^4 + 3x^2$ at $a = -1$.

Solution.

Problem 105 (Exercise 3, page 252)**.** Find the linearization $L(x)$ of the function $f(x) = \cos x$ at $a = \frac{\pi}{2}$.

Solution.

Problem 106 (Exercise 7, page 252)**.** Verify the approximation

$$\frac{1}{(1+2x)^4} \approx 1 - 8x$$

at $a = 0$. Use the approximation to approximate $\frac{1}{(1+2(0.01))^4}$

Solution.

Problem 107 (Exercise 25, page 252)**.** Use a linear approximation to estimate

$$(8.06)^{\frac{2}{3}}.$$

Solution.

Chapter 4

Applications of Differentiation

4.1 Maximum and Minimum Values

Definition 108. A function f has an *absolute maximum* (or *global maximum*) at c if $f(c) \geq f(x)$ for x in the domain of f. The number $f(c)$ is called the *maximum value* of f. Similarly, f has an *absolute minimum* at x if $f(c) \leq f(x)$ for all x in the domain of f and the number $f(c)$ is called the *minimum value* of f.

The maximum and minimum values of f are called the *extreme values* of f.

Definition 109. A function has a *local maximum* (or *relative maximum*) at c if $f(c) \geq f(x)$ when x is near c. Similarly, f has a *local minimum* (or *relative minimum*) at x if $f(c) \leq f(x)$ when x is near c.

Note that *near* a point is within some open interval containing that point.

Example 110.

- $f(x) = x^2$

- $f(x) = x^3$

- $y = 3x^4 - 16x^3 + 18x^2$ on $-1 \le x \le 4$

Theorem 111 (The Extreme Value Theorem). *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

If the conditions in the theorem requiring continuity and a closed interval are loosed then the function need not possess extreme values.

How does one find these extreme values?

Theorem 112 (Fermat's Theorem). *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

This does not mean that every a for which $f'(a) = 0$ gives an extreme value at $f(a)$, but just that those values are candidates for being extreme values.

Definition 113. A *critical number* of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 114 (Alternate phrasing of Fermat's Theorem). *If f has a local maximum or minimum at c, then c is a critical number of f.*

Problem 115 (Exercise 37, page 278). Find the critical values for $h(t) = t^{\frac{3}{4}} - 2t^{\frac{1}{4}}$.

Solution.

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).

2. Find the values of f at the endpoints of the interval.

3. The largest of the values is the absolute maximum value. The smallest of the values is the absolute minimum value.

Problem 116 (Exercise 52, page 278)**.** Find the absolute maximum and absolute minimum values of $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

Solution.

4.2 The Mean Value Theorem

Theorem 117 (Rolle's Theorem)**.** *Let $f(x)$ be a function that satisfies:*

1. *f is continuous on the closed interval $[a, b]$.*

2. *f is differentiable on the open interval (a, b).*

3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Problem 118 (Exercise 1, page 285)**.** Verify that $f(x) = 5 - 12x + 3x^2$ on the interval $[1, 3]$ satisfies the three conditions of Rolle's Theorem.

Solution.