The Closed Interval Method
To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values is the absolute maximum value. The smallest of the values is the absolute minimum value.

Problem 116 (Exercise 52, page 278)**.** Find the absolute maximum and absolute minimum values of $f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

Solution.

4.2 The Mean Value Theorem

Theorem 117 (Rolle's Theorem)**.** *Let $f(x)$ be a function that satisfies:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

Problem 118 (Exercise 1, page 285)**.** Verify that $f(x) = 5 - 12x + 3x^2$ on the interval $[1, 3]$ satisfies the three conditions of Rolle's Theorem.

Solution.

Problem 119 (Exercise 5, page 285)**.** Let $f(x) = 1 - x^{\frac{2}{3}}$. Consider the interval $[-1, 1]$. Why are the hypotheses of Rolle's Theorem not satisfied?

Solution.

Note that we can use Rolle's Theorem to show that there is exactly one root to an equation. We do this by first showing there is at least one by the Intermediate Value Theorem and then by assuming that there are two roots a and b such that $f(a) = f(b)$ and using Rolle's Theorem to show that it is not possible.

Theorem 120 (The Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 121 (Exercise 11, page 286)**.** Consider $f(x) = 3x^2 + 2x + 5$ on the interval $[-1, 1]$. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution.

2.6 Limits at Infinity; Horizontal Asymptotes

Definition 122. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Definition 123. The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L.$$

Example 124 (Horizontal asymptotes of $\arctan x$)**.**

Problem 125. Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Solution.

Problem 126. Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

Solution.

Problem 127 (Exercise 25, page 141)**.** Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+x}-3x}{1}.$$

Solution.

4.3 How Derivatives Affect The Shape Of A Graph

What does f' say about f

Definition 128. A function f is called *increasing* on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in the interval. A function f is called *decreasing* on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in the interval.

Theorem 129 (Increasing/Decreasing Test).

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Proof. On page 287 of the textbook. Uses the Mean Value Theorem.

Theorem 130 (The First Derivative Test). *Suppose that c is a critical number of a continuous function f*

- *If f' changes from positive to negative at c, then f has a local maximum at c.*
- *If f' changes from negative to positive at c, then f has a local minimum at c.*
- *If f' does not change sign at c, then f has no local maximum or minimum.*

Problem 131. Given $f(x) = x^4 - 2x^2 + 3$ find the intervals on which f is increasing or decreasing and find the local maximum and minimum values of f.

Solution.

What does f'' say about f

Definition 132. If the graph of f lies above all its tangents on an interval I, then it is called *concave upward* on I. If the graph of f lies below all its tangents on an interval I, then it is called *concave downward* on I.

Theorem 133 (Concavity Test)**.**

- *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*
- *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

Definition 134. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or vice versa.

Theorem 135 (The Second Derivative Test)**.** *Suppose f'' is continuous near c.*

- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Example 136. Back to our previous example $f(x) = x^4 - 2x^2 + 3$, let's find the intervals of concavity and the inflection points. We already found $f' = 4x^3 - 4x$.

Solution. Now we find that $f'' = 12x^2 - 4$.

Problem 137 (Exercise 25, page 295)**.** Sketch the graph of a function that satisfies these conditions:

- $f'(0) = f'(2) = f'(4) = 0$
- $f'(x) > 0$ if $x < 0$ or $2 < x < 4$
- $f''(x) > 0$ if $1 < x < 3$
- $f''(c) < 0$ if $x < 2$ or $x > 3$

Solution.

4.4 Indeterminate Forms and L'Hospitâl's Rule

If we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ (as $x \to a$), then this limit may or may not exist and is called an indeterminate form of the type $\frac{0}{0}$.

We have already seen limits of these types in Chapter 2. Some were limits of functions with removable discontinuities which we resolve by canceling common factors. Also, there was the special limit $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$.

If we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ (as $x \to a$), then this limit may or may not exist and is called an indeterminate form of the $\frac{\infty}{\infty}$.

In Section 2.6 we could evaluate this type of limit for rational functions, by dividing the numerator and the denominator by the highest power of x that occurs in the denominator. However, some of these limits of indeterminate forms we still can not solve.

Theorem 138 (L'Hospitâl's Rule)**.** *Suppose that in an interval containing a, the functions f and g are differentiable and $g'(x) \neq 0$ (except possibly at a). Supposed that*

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to z} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to z} g(x) = \pm\infty.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists.

Note. It is very important to check that the functions satisfy the conditions of L'Hospitâl's Rule before using it.

Problem 139. Find $\lim_{x\to 1} \frac{\ln(x^2)}{x-1}$.

Solution.

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it isn't clear what the value of

$$\lim_{x\to a} f(x)g(x)$$

will be, or even if it exists.

This kind of limit is called an indeterminate form of the type $0 \cdot \infty$.

We can deal with this type by writing this product $f \cdot g$ as a quotient.

$$f \cdot g = \frac{f}{1/g} = \frac{1/f}{g}.$$

Problem 140. Evaluate

$$\lim_{x\to 0} x \ln(x^2).$$

Solution.

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an indeterminate form of the type $\infty - \infty$.

To deal with this, we turn the difference into a quotient (by factoring or rationalizing or using a common denominator).

Problem 141. Find

$$\lim_{x\to 0} (\cot x - \frac{1}{x}).$$

Solution.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}.$$

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ (type 0^0)
- $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ (type ∞^0)
- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ (type 1^∞)

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{letting } y = [f(x)]^{g(x)} \text{ then } \ln y = g(x) \ln f(x);$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}.$$

Problem 142 (Exercise 63, page 305)**.** Find

$$\lim_{x \to 0^+} (cos(x))^{\frac{1}{x^2}}.$$

Solution.

4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve

A. Domain - the set of values of x for which $f(x)$ is defined

B. Intercepts - the y-intercept is $f(0)$; the x-intercepts: solve $f(x) = 0$ for x

C. Symmetry - even and odd functions

D. Asymptotes - horizontal and vertical

E. Intervals of Increasing/Decreasing - solve for critical numbers, then check intervals

F. Local maxima and minima - use the first derivative test with the critical numbers

G. Concavity and Points of Inflection - compute $f''(x)$ and use concavity test

H. Sketch the graph - Give'r

Problem 143 (Exercise 12, page 314)**.** Use these guidelines to sketch the curve

$$f(x) = \frac{x}{x^2 - 9}.$$

Solution.