Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it isn't clear what the value of

$$\lim_{x\to a} f(x)g(x)$$

will be, or even if it exists.

This kind of limit is called an indeterminate form of the type $0 \cdot \infty$.

We can deal with this type by writing this product $f \cdot g$ as a quotient.

$$f \cdot g = \frac{f}{1/g} = \frac{1/f}{g}.$$

Problem 140. Evaluate

$$\lim_{x\to 0} x \ln(x^2).$$

Solution.

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an indeterminate form of the type $\infty - \infty$.

To deal with this, we turn the difference into a quotient (by factoring or rationalizing or using a common denominator).

Problem 141. Find

$$\lim_{x\to 0} (\cot x - \frac{1}{x}).$$

Solution.

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}.$$

- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ (type 0^0)
- $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ (type ∞^0)
- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ (type 1^∞)

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{letting } y = [f(x)]^{g(x)} \text{ then } \ln y = g(x)\ln f(x);$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}.$$

Problem 142 (Exercise 63, page 305)**.** Find

$$\lim_{x\to 0^+}(cos(x))^{\frac{1}{x^2}}.$$

Solution.

4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve

A. Domain - the set of values of x for which $f(x)$ is defined

B. Intercepts - the y-intercept is $f(0)$; the x-intercepts: solve $f(x) = 0$ for x

C. Symmetry - even and odd functions

D. Asymptotes - horizontal and vertical

E. Intervals of Increasing/Decreasing - solve for critical numbers, then check intervals

F. Local maxima and minima - use the first derivative test with the critical numbers

G. Concavity and Points of Inflection - compute $f''(x)$ and use concavity test

H. Sketch the graph - Give'r

Problem 143 (Exercise 13, page 314)**.** Use these guidelines to sketch the curve

$$f(x) = \frac{x}{x^2 + 9}.$$

Solution.

[lots of workings]

Problem 144 (Exercise 26, page 314)**.** Use these guidelines to sketch the curve

$$y = \frac{x}{\sqrt{x^2 - 1}}.$$

Solution.

[lots of workings]

4.7 Optimization Problems

Similar to related rates, these are the word problems corresponding to questions about maximum and minimum values of "real life" functions.

Steps in Solving Optimization Problems See text, page 322.

Problem 145 (Exercise 10, page 328)**.** A box with an open top is to be constructed from a square piece of cardboard, 3ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

Solution.

[lots of workings]

Problem 146 (Exercise 5, page 328)**.** Find the dimensions of a rectangle with perimeter 100m whose area is as large as possible.

Solution.

[lots of workings]

Example 147 (Exercise 17, page 328)**.** Find the point on the line $y = 4x + 7$ that is closest to the origin (the point $(0, 0)$).

Solution.

4.9 Antiderivatives

In this section, we go backwards.

Definition 148. A function F is called an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 149. Consider the function

$$f(x) = 3x^2.$$

From experience, we recognize $f(x)$ as the derivative of x^3. However, the function

$$F(x) = x^3 + c$$

for any c is also an antiderivative of $f(x)$, as the derivative of any constant is 0.

Theorem 150. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + c,$$

where c is an arbitrary constant.

Remark. This is true even if the function F is some other function plus a constant, as that constant plus an arbitrary constant acts just like an arbitrary constant.

Problem 151. Find the most general antiderivative of

$$f(x) = \frac{1}{2}x^2 - 2x + 6.$$

Solution.

Problem 152. Find the most general antiderivative of

$$f(x) = \cos(x).$$

Solution.

Problem 153. Find the most general antiderivative of

$$f(x) = x^n (n \neq 1).$$

Solution.

function	particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n(n \neq 1)$	$\frac{x^n+1}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$

Figure 4.1: Some particular antiderivatives (if $F' = f$ and $G' = g$)

Problem 154. Find f if

$$f' = \sqrt{x}(6+5x),\ \ f(1) = 10.$$

Solution.

Antidifferentiation is particularly useful in analyzing motion of an object moving in a straight line. Recall if the object has position function $s(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is the antiderivative of the velocity function. Also, the acceleration function is $a(t) = v'(t)$, which means that the velocity function is the antiderivative of the acceleration function.

Problem 155 (Exercise 60, page 346)**.** A particle is moving with

$$a(t) = \cos(t) + \sin(t),\ s(0) = 0,\ v(0) = 5.$$

Find the position of the particle.

Solution.

Chapter 5

Integrals

5.1 Areas and Distances

The Area Problem

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

[picture]

We know how to find the areas of regions with straight sides such as triangles, rectangles and other polygons. How do we find the area of other regions? We take what we already know and use it as best we can: we approximate areas by polygons, usually many rectangles.

Example 156. Consider the area under the curve $y = x^2$.

[lots of space]

Problem 157. Estimate the area under the graph of $f(x) = 2 - x^2$ from $x = 0$ to $x = 1$ using four approximating rectangles and right endpoints and then left endpoints.

Solution.

[lots of space]

In both of these examples it is fairly easy to see that more and more rectangles would give better and better approximations. See the text for nice pictures.

In the tangent problem, which lead to our definition of derivative, we found ourselves looking at secant line approximations that were based on points that were closer and closer to where our line was to be tangent. In the area problem, we find ourselves looking at approximations based on increasing numbers of rectangles. Again, we will get a definition based on limits. We formalize this below.