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On Localization and Stabilization for Factorization Systems*

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Abstract. If $(\mathcal{E}, \mathcal{M})$ is a factorization system on a category $\mathcal{C}$, we define new classes of maps as follows: a map $f\colon A \to B$ is in $\mathcal{E}'$ if each of its pullbacks lies in $\mathcal{E}$ (that is, if it is *stably* in $\mathcal{E}$), and is in $\mathcal{M}^*$ if some pullback of it along an effective descent map lies in $\mathcal{M}$ (that is, if it is *locally* in $\mathcal{M}$). We find necessary and sufficient conditions for $(\mathcal{E}', \mathcal{M}^*)$ to be another factorization system, and show that a number of interesting factorization systems arise in this way. We further make the connexion with Galois theory, where $\mathcal{M}^*$ is the class of *coverings*; and include self-contained modern accounts of factorization systems, descent theory, and Galois theory.

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1. Introduction

1.1. Although many categories are richly endowed with factorization systems, there has been little investigation of general processes that produce such systems. It is of course well known that every full reflective subcategory $\mathcal{X}$ of a category $\mathcal{C}$ gives rise (under very mild conditions on $\mathcal{C}$) to a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$, where $\mathcal{E}$ is the class of maps inverted by the reflexion $I\colon \mathcal{C} \to \mathcal{X}$; the factorization systems that arise thus, called the *reflective* factorization systems, are those for which $g \in \mathcal{E}$ whenever $fg \in \mathcal{E}$ and $f \in \mathcal{E}$; and their properties were examined in considerable detail in [6]. What we investigate in the present article

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is a process that can be applied to any factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$, and then *sometimes* produces a new factorization system $(\mathcal{E}', \mathcal{M}^*)$.

Starting from the given factorization system $(\mathcal{E}, \mathcal{M})$, we define as follows new classes $\mathcal{E}'$ and $\mathcal{M}^*$ of maps: a map $f\colon A \to B$ lies in $\mathcal{E}'$ if every pullback of f lies in $\mathcal{E}$, while f lies in $\mathcal{M}^*$ if some pullback of f along an effective descent map $p\colon E \to B$ lies in $\mathcal{M}$. Thus $\mathcal{E}'$ consists of the maps *stably in* $\mathcal{E}$, and $\mathcal{M}^*$ of those *locally in* $\mathcal{M}$. When $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system, it may be said to arise by simultaneously *stabilizing* $\mathcal{E}$ *and localizing* $\mathcal{M}$. (This process is vacuous when $\mathcal{E}$ is already pullback-stable, so that $\mathcal{E}' = \mathcal{E}$; for then in fact $\mathcal{M}^* = \mathcal{M}$ – although this is not quite obvious.)

It is always the case that a map $g \in \mathcal{E}'$ and a map $n \in \mathcal{M}^*$ have the "unique diagonal fill-in property", denoted below by $g \downarrow n$; so that $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every map f has a factorization $f = ng$ with $n \in \mathcal{M}^*$ and $g \in \mathcal{E}'$. In general this is not the case; counter-examples show that it may be, in a certain sense, "far from true". However it is indeed the case in a number of interesting examples; and among the factorization systems arising thus as $(\mathcal{E}', \mathcal{M}^*)$, starting from a reflective factorization system $(\mathcal{E}, \mathcal{M})$, are several important ones: Eilenberg's monotone-light factorization for maps of compact Hausdorff spaces; the factorization of a field extension into a separable one and a purely-inseparable one; and various factorization systems associated with hereditary torsion theories on abelian categories. Our chief aims are to give a necessary and sufficient condition, in terms of the factorization system $(\mathcal{E}, \mathcal{M})$, for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system, and to work out in detail the examples above.

At the same time we wish to point out the connexion with Galois theory, as formulated in the papers [12], [13], and [14] of Janelidze: when $(\mathcal{E}, \mathcal{M})$ is the reflective factorization system arising from an *admissible* reflexion of $\mathcal{C}$ onto a full subcategory $\mathcal{X}$, the maps $f\colon A \to B$ in $\mathcal{M}^*$ are just what are called in Galois theory the *coverings* of B (or in some contexts the *central extensions* of B). So each of our positive examples is in fact a theorem about the corresponding "Galois theory". Moreover some of these Galois theories are new – certainly that corresponding to Eilenberg's factorization, which may be seen as a Galois theory for C^*-algebras.

Since this program takes us not only into very diverse areas of mathematics, but even into areas of category theory – factorization systems, descent theory, Galois theory – that will be unfamiliar to many, we have thought it best to ease the reader's task by keeping the article almost completely self-contained, revising as we go both the category theory and the background to the examples.

The occasion of our coming together for this joint investigation was Carboni's interest in the situation he discusses in Sections 9 and 10 of his forthcoming [4]. There, with $(\mathcal{E}, \mathcal{M})$ derived from a full reflective subcategory $\mathcal{X}$ of $\mathcal{C}$ – not indeed as the corresponding reflective factorization system, but as a close relation – he argues that a map $f\colon A \to B$ in $\mathcal{M}^*$ is the appropriate abstraction of "a family

$(A_b)_{b\in B}$ of objects of $\mathcal{X}$ indexed by the object B of $\mathcal{C}$ ", in connexion with his concept of "$\mathcal{C}$-completeness" of $\mathcal{X}$. However that investigation and this have proceeded quite independently; and, while our results are very relevant to his concerns, we have put particular emphasis on factorization systems, while he expresses interest rather in the more general "reflective subfibrations" that we shall mention below.

We turn to a brief description of the contents.

1.2. The modern notion of a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{C}$ was introduced by Freyd and Kelly [8]; the earlier *bicategorical structures* of Isbell [10] can be seen as those factorization systems for which every $\mathcal{E}$ is an epimorphism and every $\mathcal{M}$ a monomorphism. Later authors have further clarified the elementary properties of factorization systems, and further simplified the axioms that describe them; since there is no easily-accessible connected account of these results, we have devoted Section 2 to a succinct modern treatment.

In particular we devote Section 2.12 to a comparison of factorization systems with the more general notion of a *fibrewise-reflective subfibration of the canonical fibration* $B \mapsto \mathcal{C}/B$. Since the $\mathcal{M}$ of a factorization system is pullback-stable, the pseudofunctor $B \mapsto \mathcal{M}/B$ gives a subfibration of the canonical one; and $\mathcal{M}/B$ is reflective in $\mathcal{C}/B$, the reflexion of $f\colon A \to B$ being $m\colon C \to B$, where $f = me$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f. (Because the reflexion $\mathcal{C}/B \to \mathcal{M}/B$ is not itself a map of fibrations unless the class $\mathcal{E}$ too is pullback-stable, we say that the subfibration is *fibrewise-reflective*, rather than *reflective*.) One may very well, however, have a fibrewise-reflective subfibration – a pullback-stable class $\mathcal{M}$ of maps, with each $\mathcal{M}/B$ reflective in $\mathcal{C}/B$ – which is not part of a factorization system $(\mathcal{E}, \mathcal{M})$; we recall the result that it is indeed part of such a factorization system precisely when $\mathcal{M}$ is closed under composition.

In Section 3, drawing chiefly on [6], we recall such properties of reflective factorization systems as we need, and in particular note that the reflexions called *semi-left-exact* in [6] are exactly those called *admissible* by Janelidze in his Galois theory. For such reflexions there is a simple characterization of the maps $f\colon A \to B$ that lie in $\mathcal{M}$: namely, f is in $\mathcal{M}$ precisely when it is the pullback of its reflexion $If\colon IA \to IB$ along the reflexion-unit $B \to IB$. These are the maps called *trivial coverings* in Galois theory; so that the maps in $\mathcal{M}^*$, called as we said "coverings", are those that *locally* are trivial coverings. In Section 3.9, we describe some factorization systems *related* to the reflective ones, including that used by Carboni in the considerations we mentioned above; we shall find that, in most of the examples, such factorization systems give the same $(\mathcal{E}', \mathcal{M}^*)$ as their simpler reflective relatives.

The notion of "locally" is given by descent theory; in Section 4 we describe the basic ideas of this briefly but fully, and augment them by quoting recent results of Reiterman, Sobral, Tholen, and Janelidze that are needed for our discussion. Then, in Section 5, for an admissible reflective $\mathcal{X}$, we state and prove the basic theorem of Galois theory, in the special case suited to our context – namely that where "covering" does indeed mean "any map in $\mathcal{M}^*$ ", and not – as it occasionally does – something like "epimorphism in $\mathcal{M}^*$ ".

In Section 6 we use descent theory to give a necessary and sufficient condition for $(\mathcal{E}', \mathcal{M}^*)$ to be a factorization system: it is that, for any $f\colon A \to B$, we can find an effective descent map $p\colon E \to B$ such that, if the $(\mathcal{E}, \mathcal{M})$-factorization of the pullback $p^*(f)$ is me, we have $e \in \mathcal{E}'$. When there are enough projectives (with respect to effective descent maps), it comes to the same thing to say that the $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ of $f\colon A \to B$ has $e \in \mathcal{E}'$ whenever B is projective. The counter-examples showing how badly this may fail are at the end, in Section 10: first, $\mathcal{M}^*/B$ may not even be reflective in $\mathcal{C}/B$; secondly, when it is reflective for each B, the class $\mathcal{M}^*$ may not be closed under composition; and thirdly, when the $\mathcal{M}^*/B$ are reflective and $\mathcal{M}^*$ is closed under composition, so that $\mathcal{M}^*$ is indeed part of a factorization system $(\mathcal{F}, \mathcal{M}^*)$, the class $\mathcal{F}$ may fail to be $\mathcal{E}'$ (being strictly larger). Note that, since $\mathcal{E}' \subset \mathcal{F} \subset \mathcal{E}$ in this last case, $\mathcal{F}$ can never be pullback-stable unless it coincides with $\mathcal{E}'$.

Finally, Sections 7, 8 and 9 give the details of the positive examples mentioned above, along with the necessary mathematical background – which it would be hard to find in the literature in the precise forms needed for our purposes. Especially in these sections, the reader will find questions that could have been pursued, but were not for lack of time – in view of the publication deadline for these articles from the 1994 European Colloquium on Category Theory at Tours.

1.3. Before ending, we wish to make public our most sincere gratitude to the organizers of that colloquium, and in particular to Pierre Damphousse, for their extraordinary kindness and their unstinting personal dedication that made the meeting such a success; as well as for their very generous financial support.

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2. Revision of Factorization Systems

2.1. The general definition of a factorization system, and the really basic results, are most easily found in Sections 2.1 and 2.2 of Freyd–Kelly [8], which we largely repeat here, augmented by a few later insights – learnt in part from such others as Bousfield [3]. It is far easier to reason about factorization systems if we first introduce the simple notion of *prefactorization system*.

Given maps p and i in our category $\mathcal{C}$, we write $p \downarrow i$ if, for every pair of maps u, v with $vp = iu$, there is a unique w (often called the *diagonal fill-in*) making commutative

$$\begin{tikzcd} {} \arrow[r, "p"] \arrow[d, "u"'] & {} \arrow[d, "v"] \arrow[dl, "w"] \\ {} \arrow[r, "i"'] & {} \end{tikzcd} \qquad . \tag{2.1}$$

For any class $\mathcal{H}$ of maps in $\mathcal{C}$ we set

$$\mathcal{H}^{\uparrow} = \{p \mid p \downarrow h \text{ for all } h \in \mathcal{H}\}, \qquad \mathcal{H}^{\downarrow} = \{i \mid h \downarrow i \text{ for all } h \in \mathcal{H}\}.$$

By a *prefactorization system* $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ is meant a pair of classes of maps having $\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$. The usual arguments about Galois connexions tell us that every class $\mathcal{H}$ gives rise to prefactorization systems $(\mathcal{H}^{\uparrow}, \mathcal{H}^{\uparrow\downarrow})$ and $(\mathcal{H}^{\downarrow\uparrow}, \mathcal{H}^{\downarrow})$. If we order prefactorization systems by setting $(\mathcal{E}, \mathcal{M}) \leqslant (\mathcal{E}', \mathcal{M}')$ whenever $\mathcal{M} \subset \mathcal{M}'$, or equivalently whenever $\mathcal{E} \supset \mathcal{E}'$, they form a (possibly large) complete lattice, with $\bigwedge(\mathcal{E}_\alpha, \mathcal{M}_\alpha) = (\mathcal{M}^{\uparrow}, \mathcal{M})$ where $\mathcal{M} = \bigcap \mathcal{M}_\alpha$; the top element has $\mathcal{E} =$ the isomorphisms, $\mathcal{M} =$ all maps.

2.2. PROPOSITION. *Let $(\mathcal{E}, \mathcal{M})$ be a prefactorization system. Then*

(a) *$\mathcal{M}$ contains the isomorphisms and is closed under composition;*
(b) *every pullback of an $\mathcal{M}$ is an $\mathcal{M}$;*
(c) *if fg is an $\mathcal{M}$ so is g, provided that f is either an $\mathcal{M}$ or a monomorphism;*
(d) *if $\alpha\colon\ F \to G\colon\ \mathcal{K} \to \mathcal{C}$ is a natural transformation with each α_K in $\mathcal{M}$, and if $\lim F$ and $\lim G$ exist, then $\lim \alpha\colon\ \lim F \to \lim G$ is in $\mathcal{M}$.*

Proof. All are easy consequences of the fact that $\mathcal{M}$ is of the form $\mathcal{H}^{\downarrow}$ for some $\mathcal{H}$.

2.3. REMARK. A class $\mathcal{M}$ of maps containing the identities and satisfying (d) above necessarily satisfies (b) and (c), and contains the isomorphisms – but need not be closed under composition; see Im and Kelly [9, Thm. 2.5].

2.4. PROPOSITION. *For a prefactorization system $(\mathcal{E}, \mathcal{M})$, the intersection $\mathcal{E} \cap \mathcal{M}$ consists of the isomorphisms.*

Proof. Take $p = i \in \mathcal{E} \cap \mathcal{M},\ u = 1,\ v = 1$ in (2.1).

2.5. PROPOSITION. *When $\mathcal{C}$ is either finitely complete or finitely cocomplete, the following properties of a prefactorization system $(\mathcal{E}, \mathcal{M})$ are equivalent:*
(a) *every $\mathcal{E}$ is an epimorphism;*
(b) *if fg is in $\mathcal{M}$, so is g;*
(c) *every equalizer is in $\mathcal{M}$;*
(d) *every coretraction is in $\mathcal{M}$.*

Proof. That (a) implies (b) and that (a) implies (c) follow easily from the fact that $\mathcal{M} = \mathcal{E}^{\downarrow}$, while it is trivial that (b) implies (d) and that (c) implies (d); it remains to show that (d) implies (a). Suppose then that we have $xe = ye$ where $x, y\colon A \to B$ and $e \in \mathcal{E}$. If $\mathcal{C}$ has binary products, set $u = \langle 1_A, x\rangle,\ v = \langle 1_A, y\rangle\colon A \to A \times B$; if instead $\mathcal{C}$ has pushouts, let u, v be the cokernel-pair of e. In both cases we have, for some t and for some s,

$$ue = ve, \quad tu = tv = 1, \quad x = su, \quad y = sv. \tag{2.2}$$

Now (d) gives a diagonal w in

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e"'] & \cdot \arrow[d, "u"] \arrow[dl, "w"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} ;$$

and $tu = tv = 1$ gives $w = 1$, so that $u = v$; whence $x = y$ by (2.2).

The test for membership of $\mathcal{H}^{\uparrow}$ often simplifies:

2.6. PROPOSITION. *If $\mathcal{C}$ admits pullbacks and $\mathcal{H}$ is stable under pullbacks, a map f lies in $\mathcal{H}^{\uparrow}$ if and only if, whenever $f = ht$ with $h \in \mathcal{H}$, there is unique diagonal s in*

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "t"'] & \cdot \arrow[d, "1"] \arrow[dl, "s"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} . \tag{2.3}$$

Proof. For an f with this latter property, consider a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd} \tag{2.4}$$

with $k \in \mathcal{H}$. We have a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "t"'] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "h"'] \arrow[d, "x"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

wherein the bottom square is a pullback and $xt = u$; and the pullback h of k lies in $\mathcal{H}$. So we have an s as in (2.3), and clearly xs provides a diagonal for (2.4). As for uniqueness, if w is any diagonal for (2.4) we have $w = x\bar{s}$ and $1 = h\bar{s}$ for some $\bar{s}$; then $x\bar{s}f = wf = u = xt$ and $h\bar{s}f = f = ht$, so that $\bar{s}f = t$. By the uniqueness of s in (2.3) we have $\bar{s} = s$ and $w = xs$.

It is convenient to record:

2.7. LEMMA. *For a composite map $f = me$, suppose that $f \downarrow m$ and $e \downarrow m$. Then m is invertible.*

Proof. Since $f \downarrow m$ there is a diagonal t in

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "e"'] & \cdot \arrow[d, "1"] \arrow[dl, "t"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd} \quad ; \tag{2.5}$$

and since we clearly have commutativity in each of

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \arrow[dl, "tm"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd} \quad , \qquad \begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \arrow[dl, "1"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd} \quad ,$$

the uniqueness of the diagonal coming from $e \downarrow m$ gives $tm = 1$. By this and (2.5), m is invertible.

2.8. Although the definition in [8] of a factorization system $(\mathcal{E}, \mathcal{M})$ required each of $\mathcal{E}$ and $\mathcal{M}$ to be closed under composition, we can in fact derive this property from an apparently weaker definition; this is a considerable simplification in practice. Let us say that a class $\mathcal{N}$ of maps is *closed under composition with isomorphisms* if $vnu \in \mathcal{N}$ whenever $n \in \mathcal{N}$ and v, u are invertible. A pair $(\mathcal{E}, \mathcal{M})$ of classes of maps in $\mathcal{C}$ is said to constitute a *factorization system* if

(i) each of $\mathcal{E}$ and $\mathcal{M}$ contains the identities and is closed under composition with isomorphisms;
(ii) every map in $\mathcal{C}$ can be written as me with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;
(iii) if $vme = m'e'u$ with $m, m' \in \mathcal{M}$ and $e, e' \in \mathcal{E}$, there is a unique w making commutative the diagram

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[r, "m"] \arrow[d, "w"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd} \quad . \tag{2.6}$$

Of course (iii) is a consequence of the following – and is in fact equivalent to it in the presence of (i), since m and e' in (2.6) could be taken to be identities:

(iii)$'$ $e \downarrow m$ whenever $e \in \mathcal{E}$ and $m \in \mathcal{M}$; that is, $\mathcal{E} \subset \mathcal{M}^{\uparrow}$ – or equivalently $\mathcal{M} \subset \mathcal{E}^{\downarrow}$.

By an easy argument, the w of (2.6) is invertible if u and v are so – and in particular if u and v are identities; this shows the extent to which an $(\mathcal{E}, \mathcal{M})$-*factorization* $f = me$ of f is unique.

2.9. PROPOSITION. *Every factorization system* $(\mathcal{E}, \mathcal{M})$ *is a prefactorization system; in particular,* $\mathcal{E}$ *and* $\mathcal{M}$ *are closed under composition.*

Proof. Since we have (iii)$'$ above, we need $\mathcal{M}^{\uparrow} \subset \mathcal{E}$, and its dual. Let $f \in \mathcal{M}^{\uparrow}$ have the $(\mathcal{E}, \mathcal{M})$-factorization $f = me$; then $f \downarrow m$, while $e \downarrow m$ by (iii)$'$; thus m is invertible by Lemma 2.7, so that $f \in \mathcal{E}$ by (i).

2.10. COROLLARY. *Factorization systems are just those prefactorization systems that satisfy* (ii) *above.*

2.11. If $(\mathcal{E}, \mathcal{M})$ is a factorization system on a category $\mathcal{C}$ that admits pullbacks, we have seen that the class $\mathcal{M}$ is stable under pullbacks; the class $\mathcal{E}$, however, is not in general stable under pullbacks – a celebrated example being that where $\mathcal{C}$ is the category of topological spaces, $\mathcal{E}$ is the class of quotient maps, and $\mathcal{M}$ is the class of continuous injections. We shall commonly write $\mathcal{E}'$ for the class of those maps every pullback of which is in $\mathcal{E}$; clearly $\mathcal{E}'$ is the greatest pullback-stable class contained in $\mathcal{E}$, and it coincides with $\mathcal{E}$ precisely when $\mathcal{E}$ is pullback-stable; of course $\mathcal{E}'$ is closed under composition. When the exterior of the commutative diagram (2.6), whose top and bottom edges are $(\mathcal{E}, \mathcal{M})$-factorizations, is a pullback, the interior squares need not in general be pullbacks – but they are so if $e' \in \mathcal{E}'$. For then if we form the commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "\bar{e}"] \arrow[d, "u"'] & \cdot \arrow[r, "\overline{m}"] \arrow[d, "\overline{w}"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

wherein the right square *is* a pullback and $\overline{m}\bar{e} = me$, the left square is a pullback too, so that $\bar{e} \in \mathcal{E}$ and $\overline{m} \in \mathcal{M}$, whence there is an isomorphism x with $xe = \bar{e}$ and $\overline{m}x = m$. In particular, *when $\mathcal{E}$ is pullback-stable*, pulling back the $\mathcal{M}$-part and the $\mathcal{E}$-part of an $(\mathcal{E}, \mathcal{M})$-factorization gives another $(\mathcal{E}, \mathcal{M})$-factorization; so that we may equally say that $(\mathcal{E}, \mathcal{M})$*-factorizations are pullback-stable*. By a further ellipsis, we shall simply call a factorization system $(\mathcal{E}, \mathcal{M})$ *stable* when $\mathcal{E}$ is pullback-stable.

2.12. If $\mathcal{M}$ is any class of maps in $\mathcal{C}$, we shall write $\mathcal{M}/B$ for that *full* subcategory of the slice category $\mathcal{C}/B$ whose objects are those $f\colon\ A \to B$ lying in $\mathcal{M}$. When $(\mathcal{E}, \mathcal{M})$ is a factorization system, it is clear that each $\mathcal{M}/B$ is reflective in $\mathcal{C}/B$: the reflexion of $f\colon\ A \to B$ is $m\colon\ C \to B$, with unit $e\colon\ A \to C$ (or rather $e\colon\ (A, f) \to (C, m)$ in $\mathcal{C}/B$), where $f = me$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f.

There are however more general pullback-stable classes $\mathcal{M}$ for which each $\mathcal{M}/B$ is reflective in $\mathcal{C}/B$. We say a few words about these, chiefly as a background for the formulation in Section 6.1 below of our central aims. Since our real concern, however, is only with factorization systems, the following analysis is not needed for our positive results; accordingly we omit various details of the proofs, referring the reader instead to Im and Kelly [9], which organizes and extends some results of MacDonald and Tholen [19] and of Tholen [23].

To give a class $\mathcal{M}$ of maps in $\mathcal{C}$ is to give (the objects of) a full subcategory $\widetilde{\mathcal{M}}$ of the arrow-category $\mathcal{C}^{\mathbf{2}}$; this full subcategory is replete if, as we shall suppose, $\mathcal{M}$ is closed under composition with isomorphisms. If a map f in $\mathcal{C}$, seen as an object of $\mathcal{C}^{\mathbf{2}}$, admits a reflexion m into $\widetilde{\mathcal{M}}$, one easily sees that the unit $(q, x)\colon\ f \to m$ of the reflexion has x invertible, and so may be taken to have the form $(q, 1)\colon\ f \to m$. Suppose henceforth that $\mathcal{C}$ has pullbacks; one goes on to show that the subcategory $\widetilde{\mathcal{M}}$ is reflective in $\mathcal{C}^{\mathbf{2}}$ if and only if (a) $\mathcal{M}$ is pullback-stable (compare Remark 2.3 above) and (b) each $\mathcal{M}/B$ is reflective in $\mathcal{C}/B$.

This situation may be expressed in the language of *fibrations*. The fibration $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ given by the codomain functor may be denoted, in the language of indexed categories, by $B \mapsto \mathcal{C}/B$; then, because $\mathcal{M}$ is pullback-stable, $B \mapsto \mathcal{M}/B$ is a *subfibration*. When each $\mathcal{M}/B$ is reflective in $\mathcal{C}/B$, we may say that this subfibration is *fibrewise reflective*; but we should call it a *reflective subfibration* only when the reflexion functors $\mathcal{C}/B \to \mathcal{M}/B$ commute with pullbacks and thus constitute a map of fibrations. Clearly the fibrewise-reflective subfibration arising as above from a factorization system $(\mathcal{E}, \mathcal{M})$ is a reflective subfibration precisely when this factorization system is stable. What we now assert is that not every fibrewise-reflective subfibration arises from a factorization system.

Suppose indeed that $\mathcal{C}$ has pullbacks and that we have such a fibrewise-reflective subfibration, given by the class $\mathcal{M}$. Let the reflexion into $\mathcal{M}/B$ of a typical $f\colon A \to B$ be $m\colon C \to B$, with unit $q\colon A \to C$ as in

$$\begin{tikzcd} A \arrow[rr, "q"] \arrow[dr, "f"'] & & C \arrow[dl, "m"] \\ & B & \end{tikzcd} . \tag{2.7}$$

To say that m is invertible here is to say that $1_B\colon B \to B$ is also a reflexion of f into $\mathcal{M}/B$; and this, by Proposition 2.6, is to say that $f \in \mathcal{M}^{\uparrow}$. Accordingly, if we write $\mathcal{E}$ for $\mathcal{M}^{\uparrow}$, we have

$$m \text{ is invertible in (2.7) if and only if } f \in \mathcal{E}. \tag{2.8}$$

We now show that the following are equivalent:

(i) $(\mathcal{E}, \mathcal{M})$ *is a factorization system*;
(ii) $\mathcal{M}$ *is closed under composition*;
(iii) *in each reflexion* (2.7), *we have* $q \in \mathcal{E}$.

Since (iii) implies (i) by Section 2.8 and (i) implies (ii) by Proposition 2.9, we have only to show that (ii) implies (iii). With the reflexion of f into $\mathcal{M}/B$ given by (2.7), let the reflexion of q into $\mathcal{M}/C$ be given by $n\colon D \to C$, with unit $r\colon A \to D$. Then, since $mn \in \mathcal{M}$ while (2.7) is the reflexion of f into $\mathcal{M}/B$, there is a unique t with $tq = r$ and $mnt = m$. From $m(nt) = m1$ and $(nt)q = nr = q = 1q$, the uniqueness clause for the reflexion gives $nt = 1$; and now from $n(tn) = n1$ and $(tn)r = tq = r = 1r$, the uniqueness clause for the second reflexion gives $tn = 1$. So n is invertible, whence $q \in \mathcal{E}$ by (2.8).

When $\mathcal{C}$ is finitely complete and finitely cocomplete, the class $\mathcal{M}$ of regular monomorphisms satisfies (a) and (b) above, but is not in general closed under composition – for instance, when $\mathcal{C}$ is the dual of the category of small categories. Here, then, is a fibrewise-reflective subfibration for which $(\mathcal{E}, \mathcal{M})$ is not a factorization system.

3. Revision of Factorization Systems Derived from Reflective Subcategories

3.1. We suppose henceforth that our category $\mathcal{C}$ is finitely complete. Let $\mathcal{X}$ be a full replete reflective subcategory of $\mathcal{C}$, the inclusion functor being $H\colon\ \mathcal{X} \to \mathcal{C}$, and the reflexion $I\colon\ \mathcal{C} \to \mathcal{X}$ with its unit $\eta\colon\ 1 \to HI$ being so chosen that the counit is an identity $IH = 1$. Where confusion is unlikely we often suppress H, writing X for HX and writing $\eta_A\colon\ A \to IA$. We define on $\mathcal{C}$ a prefactorization system $(\mathcal{E}, \mathcal{M})$ by setting

$$\mathcal{E} = (H(\operatorname{mor}\mathcal{X}))^{\uparrow}, \qquad \mathcal{M} = (H(\operatorname{mor}\mathcal{X}))^{\uparrow\downarrow}; \tag{3.1}$$

note that, because $f \downarrow Hg$ if and only if $If \downarrow g$, we have (as was pointed out in [8, Lemma 4.2.1])

$$f \in \mathcal{E} \text{ if and only if } If \text{ is invertible;} \tag{3.2}$$

from which it follows that such a prefactorization system satisfies

$$\text{if} \quad e \in \mathcal{E} \quad \text{and} \quad ef \in \mathcal{E} \quad \text{then} \quad f \in \mathcal{E}. \tag{3.3}$$

Observe that in particular we have

$$\eta_A\colon\ A \to IA \quad \text{lies in } \mathcal{E}. \tag{3.4}$$

Whenever $\mathcal{C}$, besides admitting finite limits, also admits arbitrary intersections of subobjects (or even of *strong* subobjects, by which we mean those represented by strong monomorphisms) – and therefore in most cases of practical interest – this prefactorization system $(\mathcal{E}, \mathcal{M})$ is actually a factorization system. A detailed study was made by Cassidy, Hébert, and Kelly [6]; the following lemma and proposition are contained in their Theorem 3.3 (which is in fact more general still, dealing with an arbitrary adjunction rather than a reflexion). First we write a diagram in $\mathcal{C}$ that we shall often refer to below – namely

$$\begin{tikzcd}
A \arrow[dr, "w"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& C \arrow[r, "u"'] \arrow[d, "v"] & IA \arrow[d, "If"] \\
& B \arrow[r, "\eta_B"'] & IB
\end{tikzcd} , \tag{3.5}$$

where $f\colon A \to B$ is any map in $\mathcal{C}$ and the square is a pullback. Note that, since *If* lies in $\mathcal{M}$ by (3.1), Proposition 2.2(b) gives

$$v \in \mathcal{M}. \tag{3.6}$$

3.2. LEMMA. *Let $f\colon A \to B$ be such that If is a coretraction in $\mathcal{C}$, and moreover such that, whenever $f = ip$ with i a strong monomorphism lying in $\mathcal{M}$, we have i invertible. Then If is invertible; that is, $f \in \mathcal{E}$.*

Proof. Since coretractions are strong monomorphisms and strong monomorphisms are stable under pullbacks, (3.6) shows that v is a strong monomorphism in $\mathcal{M}$. By the hypothesis on f, the map v is invertible; so that we may as well take v in (3.5) to be 1 and w to be f. The square and the upper triangle in (3.5) now read $\eta_B = If.u$ and $uf = \eta_A$; applying I we get $1 = If.Iu$ and $Iu.If = 1$, so that *If* is indeed invertible.

3.3. PROPOSITION. *Let $\mathcal{C}$ admit, besides finite limits, all intersections of strong subobjects. Then the prefactorization system $(\mathcal{E}, \mathcal{M})$ is a factorization system, with the factorization of $f\colon A \to B$ constructed as follows. Having formed* (3.5) *by pulling back, write $i\colon D \to C$ for the intersection of all those strong subobjects of C, lying in $\mathcal{M}$, through which w factorizes; then $w = ig$ for some g. Now $vi \in \mathcal{M}$ and $g \in \mathcal{E}$, so that $f = (vi)g$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f.*

Proof. Since $v \in \mathcal{M}$ by (3.6) and $i \in \mathcal{M}$ by Proposition 2.2(d), we have $vi \in \mathcal{M}$ by Proposition 2.2(a); it remains only to show that $g \in \mathcal{E}$. Since applying I to the top triangle of (3.5) gives $Iu.Ii.Ig = Iu.Iw = I\eta_A = 1$, the map Ig is a coretraction; by virtue of its construction, therefore, g lies in $\mathcal{E}$ by Lemma 3.2.

3.4. We continue by recalling some further observations from [6]. For a category $\mathcal{C}$ satisfying the conditions of Proposition 3.3, the process assigning to $\mathcal{X}$ the factorization system $(\mathcal{E}, \mathcal{M})$ given by (3.1) is a *functor* Φ from the ordered set $\mathbf{R}$ of all full replete reflective subcategories of $\mathcal{C}$ (ordered by inclusion) to the ordered set $\mathbf{F}$ of all factorization systems on $\mathcal{C}$. In the other direction we have a functor $\Psi\colon \mathbf{F} \to \mathbf{R}$, sending a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ to the full replete subcategory $\mathcal{Y}$ of $\mathcal{C}$ determined by those objects Y for which the map $Y \to 1$ into the terminal object lies in $\overline{\mathcal{M}}$; that $\mathcal{Y}$ is reflective in $\mathcal{C}$ follows from Section 2.11 above, since $\mathcal{Y}$ may be identified with the slice category $\overline{\mathcal{M}}/1$ and $\mathcal{C}$ with the slice category $\mathcal{C}/1$; if the $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$-factorization of $A \to 1$ is $A \to \bar{A} \to 1$, the reflexion of A into $\mathcal{Y}$ is $\bar{A}$, with unit $A \to \bar{A}$. In fact Ψ is the right adjoint to Φ, for we have

$$\Psi\Phi = 1, \qquad \Phi\Psi \leqslant 1. \tag{3.7}$$

To see the first of these observe that, when $\Phi\mathcal{X} = (\mathcal{E}, \mathcal{M})$, the map $\eta_A\colon A \to IA$ lies in $\mathcal{E}$ by (3.4) while the map $IA \to 1$ lies in $\mathcal{M}$ by (3.1) – since the terminal object 1 surely lies in the reflective $\mathcal{X}$. The $(\mathcal{E}, \mathcal{M})$-factorization of $A \to 1$ thus being $A \to IA \to 1$, we see that $A \to 1$ lies in $\mathcal{M}$ precisely when $A \to IA$ is invertible; that is, precisely when $A \in \mathcal{X}$.

As for the second assertion of (3.7), write $\mathcal{X}$ for $\Psi(\overline{\mathcal{E}}, \overline{\mathcal{M}})$, write $(\mathcal{E}, \mathcal{M})$ for $\Phi\mathcal{X}$, and consider for a map $f\colon A \to B$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & IA \arrow[r, "u"] \arrow[d, "If"] & 1 \arrow[d, "1"] \\ B \arrow[r, "\eta_B"'] & IB \arrow[r, "v"'] & 1 \end{tikzcd} \quad ;$$

by the description above of the reflexion onto $\Psi(\overline{\mathcal{E}}, \overline{\mathcal{M}})$, we have $\eta_A, \eta_B \in \overline{\mathcal{E}}$ and $u, v \in \overline{\mathcal{M}}$. Consequently we have $If \in \overline{\mathcal{M}}$ by Proposition 2.2(b), so that If is invertible precisely when it belongs to $\overline{\mathcal{E}}$. By Proposition 2.2(b) again, this time in the dual form, $If \in \overline{\mathcal{E}}$ if and only if $If.\eta_A \in \overline{\mathcal{E}}$; that is, if and only if $\eta_B f \in \overline{\mathcal{E}}$. Since $\mathcal{E}$ consists of those f with If invertible, we have

$$f \in \mathcal{E} \quad \text{if and only if} \quad \eta_B f \in \overline{\mathcal{E}}. \tag{3.8}$$

Certainly, then, $f \in \mathcal{E}$ if $f \in \overline{\mathcal{E}}$; so that $(\mathcal{E}, \mathcal{M}) \leqslant (\overline{\mathcal{E}}, \overline{\mathcal{M}})$, as asserted in (3.7).

Accordingly, Φ identifies the ordered set $\mathbf{R}$ with a full and coreflective sub-ordered-set of $\mathbf{F}$. We have seen that any factorization system in the image of Φ satisfies (3.3); and in fact the converse is also true. For if $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ satisfies (3.3), we see from (3.8) that $\mathcal{E} \subset \overline{\mathcal{E}}$; which gives $(\overline{\mathcal{E}}, \overline{\mathcal{M}}) = \Phi\Psi(\overline{\mathcal{E}}, \overline{\mathcal{M}})$. Such factorization systems are said to be *reflective*. The example where $\mathcal{C}$ is the category of sets, $\mathcal{E}$ is the surjections, and $\mathcal{M}$ is the injections, shows that not every factorization system is reflective.

3.5. Consider again the prefactorization system $\Phi\mathcal{X} = (\mathcal{E}, \mathcal{M})$ defined as in (3.1) from a full replete reflective subcategory $\mathcal{X}$ of the finitely-complete $\mathcal{C}$, and the diagram (3.5) wherein the square is a pullback. It may happen that $w \in \mathcal{E}$ for every map f in $\mathcal{C}$: when this is so, $f = vw$ is already the $(\mathcal{E}, \mathcal{M})$-factorization of f, and $(\mathcal{E}, \mathcal{M})$ is a factorization system on $\mathcal{C}$ regardless of whether $\mathcal{C}$ admits intersections of strong subobjects. In such cases the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ was said in [6] to be *simple*; Theorem 4.1 of [6] includes, among other things, the equivalence of the following two conditions:

(a) *the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ is simple*;
(b) *a map $f\colon A \to B$ in $\mathcal{C}$ lies in $\mathcal{M}$ if and only if the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & IA \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{3.9}$$

is a pullback.

Proof. Given (a) we have to prove the “only if” part of (b), since the “if” part is automatic by Proposition 2.2(b); in other words, we are to prove that w is invertible in (3.5) when $f \in \mathcal{M}$. But $w \in \mathcal{M}$ when $f \in \mathcal{M}$ by (3.6) and Proposition 2.2(c); so that w is invertible since $w \in \mathcal{E}$ by (a).

Turning to the converse, we are, for a general $f\colon A \to B$, to show that $w \in \mathcal{E}$ in (3.5). Consider the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "1"'] & IC \arrow[d, "Iu"] \\ C \arrow[r, "u"'] \arrow[d, "v"'] & IA \arrow[d, "If"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \quad ;$$

the bottom square is a pullback by the construction of (3.5), while the exterior is a pullback by (b) since $v \in \mathcal{M}$ by (3.6) and since applying I to the bottom square gives $If.Iu = Iv$; therefore the top square too is a pullback. However $Iu.Iw.u = u.1_C$, since $Iu.Iw = 1$ by applying I to the top triangle of (3.5); so that the pullback property of the top square above gives $Iw.u = \eta_C$. Applying I now gives $Iw.Iu = 1$, establishing the invertibility of Iw.

3.6. The reflexion of $\mathcal{C}$ onto $\mathcal{X}$ was said in [6] to be *semi-left-exact* if every pullback diagram

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "v"'] & X \arrow[d, "g"] \\ B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{3.10}$$

with $X \in \mathcal{X}$ has $u \in \mathcal{E}$. Clearly a *semi-left-exact reflexion is simple*; for then in (3.5) we have $u \in \mathcal{E}$, giving $w \in \mathcal{E}$ by (3.3) since $\eta_A \in \mathcal{E}$ by (3.4). Example 4.4 of [6] shows that *a simple reflexion need not be semi-left-exact*; while Theorem 4.3 of [6] gives two further conditions equivalent to semi-left-exactness – namely (i) *I preserves the pullback of f and g if $g \in \mathcal{M}$*; (ii) *every pullback of an $\mathcal{E}$ by an $\mathcal{M}$ is an $\mathcal{E}$*. The reader will find it easy to reconstruct the proofs of these equivalences.

In his work on a general categorical version of Galois theory, part of which we shall revise in Section 5 below, Janelidze approached the matter of a reflective full subcategory $\mathcal{X}$ of $\mathcal{C}$ from a different angle. For each $B \in \mathcal{C}$ we have a functor $I^B\colon\ \mathcal{C}/B \to \mathcal{X}/IB$ sending $f\colon\ A \to B$ to If; and this functor has a right adjoint $H^B\colon\ \mathcal{X}/IB \to \mathcal{C}/B$ sending $g\colon\ X \to IB$ to its pullback v as in (3.10). To ask that each H^B be fully faithful, or equivalently that the counit $I^B H^B \to 1$ be invertible, is precisely to ask that u belong to $\mathcal{E}$ in (3.10), and thus to ask the reflexion to be semi-left-exact. The full fidelity of H^B being important for the Galois theory, Janelidze [12] called reflexions with this property *admissible* – a usage later followed by Janelidze and Kelly in their study [15] of central extensions.

It would accordingly seem that *admissiblity* and *semi-left-exactness* are two names for the same thing. However Janelidze's Galois theory involves not only a full reflective subcategory $\mathcal{X}$ of $\mathcal{C}$, but also a pullback-stable class Θ of maps in $\mathcal{C}$; his $\mathcal{C}/B$ denotes not the slice category itself, but the full subcategory of this whose objects are the maps in Θ; and thus his admissibility requires that u belong to $\mathcal{E}$ in the pullback (3.10) *only* when $g \in \Theta$. The reflexion of the category of groups onto that of abelian groups is (see [15, Theorem 3.4]) admissible when Θ consists of the surjections, but not when it consists of all maps (take B in (3.10) to be the symmetric group on three elements, and X to be the identity group 1). For the work on central extensions in [15], Θ was in fact the surjections;

but in many applications of Galois theory (for example to field extensions, or to covering spaces) Θ consists of all maps. Since we restrict ourselves to this latter case in the present article, we may use “admissible” for “semi-left-exact”.

3.7. The reflexion of $\mathcal{C}$ onto $\mathcal{X}$ was said in [6] to *have stable units* if *every* pullback of each unit $\eta_B\colon\ B \to IB$ lies in $\mathcal{E}$; that is, if u in the pullback (3.10) lies in $\mathcal{E}$ whether or not $X \in \mathcal{X}$. That this condition is *strictly* stronger than admissibility was shown in [6, Example 4.6]. We leave it to the reader to verify that having stable units is equivalent to *the preservation by I of all those pullbacks*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "k"] \\ B \arrow[r, "h"'] & Z \end{tikzcd} \tag{3.11}$$

in $\mathcal{C}$ for which $Z \in \mathcal{X}$.

3.8. Stronger still is the condition that I preserve *all* pullbacks; which, since it trivially preserves 1, is to say that it is left exact. As was observed in [6, Example 4.9], the reflexion of abelian groups onto the torsion-free ones has stable units, but is not left exact. In this case of left-exact I, the reflective $\mathcal{X}$ is said to be a *localization* of $\mathcal{C}$. As was shown in [6, Theorem 4.7], the left-exactness of I is equivalent to the condition that *every pullback of an $\mathcal{E}$ be an $\mathcal{E}$*. So the localizations of $\mathcal{C}$ correspond to those factorization systems $(\mathcal{E}, \mathcal{M})$ which are both *reflective* (that is, satisfy (3.3)) and *stable* (in the sense of Section 2.11); these have been called the *local* factorization systems.

3.9. It may sometimes be convenient below to consider on $\mathcal{C}$ a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ which is not a reflective one, but is related to a (full, replete) reflective subcategory $\mathcal{X}$ of $\mathcal{C}$ by the fact that $\overline{\mathcal{E}}$ consists of those maps inverted by the reflexion I which *lie in a certain class $\mathcal{F}$ of epimorphisms*. The following observations on this situation generalize [6, Proposition 5.5] – and improve it, in the sense that they require of $\mathcal{C}$ only finite limits, and not general intersections of subobjects.

Suppose, then, that $\mathcal{X}$ is a reflective subcategory as in Section 3.1, and that $(\mathcal{E}, \mathcal{M}) = \Phi\mathcal{X}$ is the prefactorization system defined by (3.1); if $\mathcal{C}$ only has finite limits, $(\mathcal{E}, \mathcal{M})$ may fail to be a factorization system. Suppose, however, that we are given on $\mathcal{C}$ a factorization system $(\mathcal{F}, \mathcal{N})$, for which $\mathcal{F}$ *is contained in the class of epimorphisms*; and consider, in the ordered set of prefactorization systems on $\mathcal{C}$, the join

$$(\overline{\mathcal{E}}, \overline{\mathcal{M}}) = (\mathcal{E}, \mathcal{M}) \vee (\mathcal{F}, \mathcal{N}), \tag{3.12}$$

observing that, by Section 2.1, we have

$$\overline{\mathcal{E}} = \mathcal{E} \cap \mathcal{F}, \qquad \overline{\mathcal{M}} \supset \mathcal{M} \cup \mathcal{N}. \tag{3.13}$$

So $\overline{\mathcal{E}}$ consists of those $f \in \mathcal{F}$ inverted by I. Now in fact:

$(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is a factorization system, and $f\colon A \to B$ lies in $\overline{\mathcal{M}}$ if and only if the map

$$\langle f, \eta_A \rangle\colon A \to B \times IA \tag{3.14}$$

lies in $\mathcal{N}$.

Proof. Given $f\colon A \to B$, form the diagram (3.5) as before. Since $v \in \mathcal{M}$, we certainly have $v \in \overline{\mathcal{M}}$. Let the $(\mathcal{F}, \mathcal{N})$-factorization of w be $w = ne$; then $n \in \overline{\mathcal{M}}$ since $n \in \mathcal{N}$, so that we have $f = (vn)e$ where $vn \in \overline{\mathcal{M}}$. The map e, being epimorphic because it lies in $\mathcal{F}$, is taken by the left adjoint I to an epimorphism Ie in $\mathcal{X}$. However I applied to the top triangle of (3.5) gives $Iu.In.Ie = Iu.Iw = I\eta_A = 1$, showing that the epimorphism Ie is a coretraction; accordingly it is invertible. Thus e lies in $\mathcal{E}$ and hence in $\mathcal{E} \cap \mathcal{F} = \overline{\mathcal{E}}$, and $f = (vn)e$ is the $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$-factorization of f. Clearly $f \in \overline{\mathcal{M}}$ precisely when e here is invertible, which is to say that $w \in \mathcal{N}$. However we have $\langle f, \eta_A \rangle = \langle v, u \rangle w$, and $\langle v, u \rangle\colon C \to B \times IA$ is an equalizer, thus lying in $\mathcal{N}$ by Proposition 2.5. We conclude from Proposition 2.2 that $w \in \mathcal{N}$ if and only if $\langle f, \eta_A \rangle \in \mathcal{N}$.

Let us write $\mathcal{W}$ for the full reflective subcategory $\overline{\mathcal{M}}/1$ of $\mathcal{C}$. Since we get the reflexion of A into $\mathcal{W}$ by taking the $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$-factorization of $A \to 1$, it follows from the above that this reflexion is given by $\rho_A\colon A \to JA$, where

$$A \xrightarrow[\rho_A]{} JA \xrightarrow[\sigma_A]{} IA$$

is the $(\mathcal{F}, \mathcal{N})$-factorization of η_A. Of course the object A lies in $\mathcal{W}$ if and only if $\eta_A \in \mathcal{N}$; and by Proposition 2.5 this is equally to say that there is some $n\colon A \to X$ with $n \in \mathcal{N}$ and $X \in \mathcal{X}$. Thus $\mathcal{W}$ may be called *the $\mathcal{N}$-hull of $\mathcal{X}$.*

The example where $\mathcal{C}$ is the category of sets, $\mathcal{X}$ is $\{1\}, \mathcal{F}$ is the surjections, and $\mathcal{N}$ is the injections shows that, in general, the factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is not reflective.

4. Revision of Descent Theory

4.1. Descent theory has a long history, going back to work of Grothendieck around 1959, and has become much better explicated and understood following the contributions of later authors; there is now available a convenient survey of the ideas in modern language by Janelidze and Tholen [17], which is to be augmented by a sequel [18]. Drawing chiefly on the first of these, we review here only the simple results we need below.

We continue to suppose that $\mathcal{C}$ is a category with finite limits. By a *precategory object P in $\mathcal{C}$*, or just a *precategory in $\mathcal{C}$* for short, is meant a diagram of the form

$$\begin{tikzcd} P_2 \arrow[r, shift left=6, "q"] \arrow[r, "m"] \arrow[r, shift right=6, "r"'] & P_1 \arrow[r, shift left=6, "d"] \arrow[r, shift right=6, "c"'] & P_0 \arrow[l, "e"'] \end{tikzcd} \tag{4.1}$$

satisfying the conditions

$$de = ce = 1, \qquad dr = cq, \qquad dm = dq, \qquad cm = cr. \tag{4.2}$$

With the obvious notion of morphism, precategories in $\mathcal{C}$ constitute a functor category [**P**, $\mathcal{C}$], where $\mathbf{P}^{\mathrm{op}}$ is the evident subcategory of the simplical category $\mathbf{\Delta}$. Every category object in $\mathcal{C}$ determines a precategory, and $\mathrm{Cat}(\mathcal{C})$ is a full subcategory of [**P**, $\mathcal{C}$]; the precategory P is a category precisely when the square represented by the second equation in (4.2) is a pullback and the "composition operation" m satisfies the associativity and unit laws; equivalently, the precategory P in $\mathcal{C}$ is a category in $\mathcal{C}$ precisely when each $\mathcal{C}(A, P)$ is a category in **Set**.

A category P in $\mathcal{C}$ is called a *groupoid*, or a *preorder*, or an *equivalence relation*, when each $\mathcal{C}(A, P)$ is so – similarly for a *group* or a *monoid*; it is easy to give elementary formulations of these properties in terms of the data above. Of course a category is an equivalence relation precisely when it is both a groupoid and a preorder. We may also call a category P a *pregroupoid* if there is some $k\colon\ P_1 \to P_2$ with $qk = 1$ and $mk = ed$. What we need from this notion is that any functor $\mathcal{C} \to \mathcal{D}$ carries pregroupoids in $\mathcal{C}$ to pregroupoids in $\mathcal{D}$, and that a category is a groupoid if and only if it is a pregroupoid.

The diagonal functor $\Delta\colon\ \mathcal{C} \to [\mathbf{P}, \mathcal{C}]$ being fully faithful, we may identify an object B of $\mathcal{C}$ with its image ΔB, which is the category

$$\begin{tikzcd} B \arrow[r, shift left=6, "1"] \arrow[r, "1"] \arrow[r, shift right=6, "1"'] & B \arrow[r, shift left=6, "1"] \arrow[r, shift right=6, "1"'] & B \arrow[l, "1"'] \end{tikzcd}, \tag{4.3}$$

it is of course an equivalence relation. A precategory P is isomorphic to such a B if and only if every map in (4.1) is invertible; whereupon we call P a *discrete category*.

4.2. A morphism $f\colon\ P \to Q$ of precategories, given by components $f_i\colon\ P_i \to Q_i$, is said to be a *discrete opfibration* if each of the squares

$$\begin{tikzcd} P_2 \arrow[r, "q"] \arrow[d, "f_2"'] & P_1 \arrow[r, "d"] \arrow[d, "f_1"] & P_0 \arrow[d, "f_0"] \\ Q_2 \arrow[r, "q"'] & Q_1 \arrow[r, "d"'] & Q_0 \end{tikzcd} \tag{4.4}$$

is a pullback. Clearly discrete opfibrations are closed under composition; g is a discrete opfibration whenever fg and f are so; and every pullback in [**P**, $\mathcal{C}$] of a discrete opfibration is a discrete opfibration. The following are easy to verify in the case $\mathcal{C} = \mathbf{Set}$, and are accordingly true – arguing with generalized elements – for any finitely-complete $\mathcal{C}$:

(a) *when P and Q are categories, the left square in* (4.4) *is a pullback if the right square is one*;
(b) *if $f\colon\ P \to Q$ is a discrete opfibration, then P is a category* [*resp. a groupoid, a preorder, an equivalence relation*] *whenever Q is so.*

Given a precategory Q, we write $\mathcal{C}^Q$ for the full subcategory of the slice category $[\mathbf{P}, \mathcal{C}]/Q$ determined by the discrete opfibrations $P \to Q$. For reasons that we shall not break off to explain here – they will be clear to those familiar with internal category theory – $\mathcal{C}^Q$ is called *the category of* (*internal*) *actions of the precategory Q in $\mathcal{C}$*; it is easy to see, for instance, that we get the usual notion of action when Q is a group or a monoid. Any morphism $g\colon\ R \to Q$ of precategories induces by pullback a functor $[\mathbf{P}, \mathcal{C}]/Q \to [\mathbf{P}, \mathcal{C}]/R$, which carries discrete opfibrations to discrete opfibrations and thus restricts to a functor

$$\mathcal{C}^g\colon\ \mathcal{C}^Q \to \mathcal{C}^R. \tag{4.5}$$

Consider $\mathcal{C}^B$ when the precategory B is just an object B of $\mathcal{C}$. Clearly every map $f\colon\ A \to B$ in $\mathcal{C}$ is a discrete opfibration; and one easily sees that a general $f\colon\ P \to B$ in $[\mathbf{P}, \mathcal{C}]$ is a discrete opfibration if and only if P is a discrete category. It follows that we have a full inclusion

$$\mathcal{C}/B \to \mathcal{C}^B \tag{4.6}$$

which is equivalence of categories.

4.3. For any map $p\colon\ E \to B$ in $\mathcal{C}$, its kernel-pair $d, c\colon\ E \times_B E \to E$ extends in an obvious way to an equivalence relation

$$\begin{tikzcd}[column sep=large] E \times_B E \times_B E \arrow[r, shift left=4, "q"] \arrow[r, "m"] \arrow[r, shift right=4, "r"'] & E \times_B E \arrow[r, shift left=4, "d"] \arrow[r, shift right=4, "c"'] & E \arrow[l, "e"'] \end{tikzcd} \tag{4.7}$$

in $\mathcal{C}$; let us denote this by $\overline{B} \in \mathrm{Cat}(\mathcal{C}) \subset [\mathbf{P}, \mathcal{C}]$. The map p admits, in $\mathrm{Cat}(\mathcal{C})$, a factorization

$$E \xrightarrow[p'']{} \overline{B} \xrightarrow[p']{} B\ , \tag{4.8}$$

where p'' and p' are the evident morphisms. (Although we do not need this, (4.8) is an internal-category-theory instance of the well-known factorization of any functor into a p'' that is bijective on objects and a p' that is fully faithful.) The map p: $E \to B$ is said to be an *effective descent map* if the functor

$$\mathcal{C}^{p'}\colon\ \mathcal{C}^{B} \to \mathcal{C}^{\overline{B}} \tag{4.9}$$

is an equivalence of categories; this is equally to say that its composite with (4.6), which we denote by

$$p^{\#}\colon\ \mathcal{C}/B \to \mathcal{C}^{\overline{B}}, \tag{4.10}$$

is an equivalence of categories.

This apparently *ad hoc* definition admits of the following explanation. The functor p^*: $\mathcal{C}/B \to \mathcal{C}/E$ given by pulling back along p has the left adjoint $p_!$ given by composition with p, so that we have a monad $T = p^* p_!$ on $\mathcal{C}/E$; the comparison functor from $\mathcal{C}/B$ to the Eilenberg-Moore category of T-algebras, and the forgetful functor from this last to $\mathcal{C}/E$, give a diagram

$$\begin{tikzcd} \mathcal{C}/B \arrow[rr, "q"] \arrow[dr, "p^*"'] & & T\text{-}Alg \arrow[dl] \\ & \mathcal{C}/E & \end{tikzcd} \tag{4.11}$$

On the other hand, the factorization $p = p'p''$ of (4.8) gives us by (4.5) the diagram

$$\begin{tikzcd} \mathcal{C}^{B} \arrow[rr, "\mathcal{C}^{p'}"] \arrow[dr, "\mathcal{C}^{p}"'] & & \mathcal{C}^{\overline{B}} \arrow[dl, "\mathcal{C}^{p''}"] \\ & \mathcal{C}^{E} & \end{tikzcd} \tag{4.12}$$

The point now is that a somewhat long but essentially straightforward calculation allows us to identify, modulo the equivalence (4.6), the diagram (4.11) with (4.12). Accordingly we conclude that p: $E \to B$ *is an effective descent map precisely when* p^*: $\mathcal{C}/B \to \mathcal{C}/E$ *is monadic*.

4.4. It will be convenient to adopt the following notation for the effect of $p^{\#}\colon \mathcal{C}/B \to \mathcal{C}^{\overline{B}}$ on a typical object $f\colon A \to B$ of $\mathcal{C}/B$. For the domain of $p^{\#}(f)$ we write $\bar{A}$, so that we have in [**P**, $\mathcal{C}$] a pullback

$$\begin{tikzcd} \bar{A} \arrow[r, "s"] \arrow[d, "p^{\#}(f)"'] & A \arrow[d, "f"] \\ \overline{B} \arrow[r, "p'"'] & B \end{tikzcd} \quad ; \tag{4.13}$$

and for the components of $p^{\#}(f)$ we write $f_i\colon A_i \to B_i$, where $i = 0, 1, 2$. Since $\overline{B}$ denotes (4.7), we have of course $B_0 = E$; and the squares of the diagram

$$\begin{tikzcd} A_2 \arrow[r, "q"] \arrow[d, "f_2"'] & A_1 \arrow[r, "d"] \arrow[d, "f_1"'] & A_0 \arrow[r, "s_0"] \arrow[d, "f_0"'] & A \arrow[d, "f"] \\ B_2 \arrow[r, "q"'] & B_1 \arrow[r, "d"'] & B_0 \arrow[r, "p"'] & B \end{tikzcd} \tag{4.14}$$

are pullbacks, the rightmost square being the 0-component of the pullback (4.13) – which we may record as

$$(p^{\#}(f))_0 = p^*(f). \tag{4.15}$$

Similarly, for the effect of $p^{\#}$ on a morphism

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

in $\mathcal{C}/B$, we write simply

$$\begin{tikzcd} \bar{A} \arrow[rr, "p^{\#}(h)"] \arrow[dr, "p^{\#}(f)"'] & & \overline{C} \arrow[dl, "p^{\#}(g)"] \\ & \overline{B} & \end{tikzcd} \quad , \tag{4.16}$$

with $h_i\colon\ A_i \to C_i$ for the components of h. Note that there is no real ambiguity in the meaning of $p^{\#}(h)$, since the diagram

$$\begin{tikzcd} \bar{A} \arrow[r, "s"] \arrow[d, "p^{\#}(h)"'] & A \arrow[d, "h"] \\ \overline{C} \arrow[r, "t"'] & C \end{tikzcd} \quad , \tag{4.17}$$

wherein t is the analogue for C of the s of (4.13), is also a pullback.

4.5. Observe that, since a monadic functor reflects isomorphisms, *a map* $f\colon A \to B$ *is invertible if its pullback along some effective descent map* $p\colon\ E \to B$ *is invertible*. We could give an alternative direct argument for this: if the pullback f_0 of f along p in (4.14) is invertible, so are the further pullbacks f_1 and f_2, whence $p^{\#}(f)$ is invertible; since $p^{\#}(f)$ is the image under the equivalence $p^{\#}$ of the map $f\colon\ f \to 1_B$ in $\mathcal{C}/B$, it follows that f is invertible.

The following consequence of this is useful:

4.6. LEMMA. *In a finitely-complete* $\mathcal{C}$*, suppose that the exterior and the left square of the commutative diagram*

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "g"'] & A \arrow[r, "t"] \arrow[d, "f"] & M \arrow[d, "h"] \\ E \arrow[r, "p"'] & B \arrow[r, "q"'] & N \end{tikzcd} \tag{4.18}$$

are pullbacks. Then the right square is a pullback if p *is an effective descent map.*

Proof. Let the pullback of h along q be

$$\begin{tikzcd} D \arrow[r, "v"] \arrow[d, "u"'] & M \arrow[d, "h"] \\ B \arrow[r, "q"'] & N \end{tikzcd} \quad , \tag{4.19}$$

and let $w\colon A \to D$ be the unique map with $vw = t$ and $uw = f$. Since the exterior of (4.18) and the diagram (4.19) are pullbacks, so too (by a classical and simple result) is

$$\begin{array}{ccc} C & \xrightarrow{ws} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ E & \xrightarrow[p]{} & B \end{array} .$$

Accordingly $p^*(w)$ is invertible in $\mathcal{C}/E$; so that w is invertible in $\mathcal{C}/B$, and hence in $\mathcal{C}$, as desired.

4.7. We need the following three results about effective descent maps:

(a) *In an exact category, the effective descent maps are precisely the regular epimorphisms.*
(b) *In any finitely complete $\mathcal{C}$, the effective descent maps are stable under pullbacks.*
(c) *In any finitely complete $\mathcal{C}$, the effective descent maps are closed under composition.*

The first, which is classical, can be found in the survey article [17]. The second was proved by Sobral and Tholen [22], and the third by Reiterman, Sobral, and Tholen [21]; these authors supposed $\mathcal{C}$ to admit coequalizers as well as finite limits, but it seems on examining their proofs that this is unnecessary. (In any case, $\mathcal{C}$ does admit coequalizers in our specific applications below.)

5. Revision of Categorical Galois Theory

5.1. We recall here the essence of the Galois theory developed by Janelidze in the articles [12, 13], and [14], restricting ourselves to the case where the class Θ mentioned in Section 3.6 above consists of all maps.

We begin with a reflexion of $\mathcal{C}$ onto $\mathcal{X}$ as in Section 3.1, and the prefactorization system $(\mathcal{E}, \mathcal{M}) = \Phi\mathcal{X}$ introduced there; but we now suppose this reflexion to be *admissible* (also called *semi-left-exact*) in the sense of Section 3.6. Thus $(\mathcal{E}, \mathcal{M})$ is a factorization system on our finitely-complete $\mathcal{C}$, whether or not general intersections of strong subobjects exist.

As we said in Section 3.6, it is precisely in this admissible case that the functor $H^B\colon \mathcal{X}/IB \to \mathcal{C}/B$ sending $g\colon X \to IB$ to its pullback along η_B is fully faithful. This pullback lies of course in the pullback-stable $\mathcal{M}$; while any $f\colon A \to B$ in $\mathcal{M}$ is such a pullback by (b) of Section 3.5. Thus the image of

H^B is just the full subcategory $\mathcal{M}/B$ of $\mathcal{C}/B$, and the restrictions of I^B and of H^B give an adjoint equivalence

$$I^B \dashv H^B\colon\ \mathcal{X}/IB \simeq \mathcal{M}/B. \tag{5.1}$$

In this context of Galois theory, the maps $f\colon\ A \to B$ that lie in $\mathcal{M}/B$ are often called *the trivial coverings of* B. More generally, if $p\colon\ E \to B$ is an effective descent map, the map $f\colon\ A \to B$ is said to be *split by* (E, p) when the pullback $p^*(f) = g$ in

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{5.2}$$

lies in $\mathcal{M}$, and so constitutes a trivial covering of E; the full subcategory of $\mathcal{C}/B$ given by all such f is denoted by $\mathrm{Spl}(E, p)$. A map $f\colon\ A \to B$ is said to be a *covering* of B if it is split by *some* effective descent map $p\colon\ E \to B$; so that the coverings constitute in $\mathcal{C}/B$ a full subcategory

$$\mathrm{Cov}B = \bigcup_p \mathrm{Spl}(E, p). \tag{5.3}$$

This is in fact a *directed* union; for if $p_1\colon\ E_1 \to B$ and $p_2\colon\ E_2 \to B$ are effective descent maps, so too by Section 4.7 above is the diagonal p of the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[r, "p_1"'] & B \end{tikzcd}, \tag{5.4}$$

and clearly each $\mathrm{Spl}(E_i, p_i)$ is contained in $\mathrm{Spl}(E, p)$. Often there is an individual effective descent map $p\colon\ E \to B$ for which $\mathrm{Cov}\ B = \mathrm{Spl}(E, p)$; this is true, for example, by an easy argument, if there is an effective descent map $p\colon\ E \to B$ for which E is projective with respect to all effective descent maps.

The goal of Galois theory is, in the first place, to describe for a given effective descent map $p\colon\ E \to B$ the category $\mathrm{Spl}(E, p)$; this in turn provides a description of $\mathrm{Cov}\ B$, whether as the union (5.3) or as an individual $\mathrm{Spl}(E, p)$. We set out as follows towards this goal.

5.2. The reflexion $I \dashv H\colon \mathcal{X} \to \mathcal{C}$ induces a reflexion

$$[\mathbf{P}, I] \longrightarrow\!\!\!\!\!\!\dashv [\mathbf{P}, H]\colon\ [\mathbf{P}, \mathcal{X}] \longrightarrow [\mathbf{P}, \mathcal{C}] \tag{5.5}$$

between the functor categories of Section 4.1, with unit $[\mathbf{P}, \eta]$; it does no harm to write simply I, H, and η for $[\mathbf{P}, I], [\mathbf{P}, H]$, and $[\mathbf{P}, \eta]$, treating (5.5) as an *extension* of the reflexion $I \dashv H\colon\ \mathcal{X} \to \mathcal{C}$, where $\mathcal{C}$ is embedded in $[\mathbf{P}, \mathcal{C}]$ by the fully-faithful diagonal $\Delta\colon\ \mathcal{C} \to [\mathbf{P}, \mathcal{C}]$, and so on. Because a pullback like (3.10) in the functor category is formed pointwise, *the reflexion* (5.5) *is again admissible*. Applying to it the functor Φ of Section 3.4 gives a reflective factorization system – which we still call $(\mathcal{E}, \mathcal{M})$ – on $[\mathbf{P}, \mathcal{C}]$. It is immediate that a map $f\colon\ P \to Q$ in $[\mathbf{P}, \mathcal{C}]$ lies in $\mathcal{M}$ if and only if each of its components $f_i\colon\ P_i \to Q_i$ lies in the class $\mathcal{M}$ of $\mathcal{C}$; similarly for $\mathcal{E}$. Accordingly, for $Q \in [\mathbf{P}, \mathcal{C}]$, the equivalence (5.1) generalizes to

$$I^Q \longrightarrow\!\!\!\!\!\!\dashv H^Q\colon\ [\mathbf{P}, \mathcal{X}]/IQ\ \simeq\ \mathcal{M}/Q. \tag{5.6}$$

Given an object $f\colon\ P \to Q$ of $\mathcal{M}/Q$, and its image If in $[\mathbf{P}, \mathcal{X}]/IQ$, we have as in (3.9) a pullback

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[d, "f"'] & IP \arrow[d, "If"] \\ Q \arrow[r, "\eta_Q"'] & IQ \end{tikzcd} . \tag{5.7}$$

Now if If is a discrete opfibration (in $[P, \mathcal{X}]$ or in $[\mathbf{P}, \mathcal{C}]$ – it makes no difference), so too is its pullback f. On the other hand, if f is a discrete opfibration, so is If; for the components f_i of f lie in $\mathcal{M}$, and (by Section 3.6) I preserves the pullback of u and v if $u \in \mathcal{M}$. Thus (5.6) restricts further to an equivalence

$$I^Q \longrightarrow\!\!\!\!\!\!\dashv H^Q\colon\ \mathcal{X}^{IQ}\ \simeq\ \mathcal{C}^Q \cap \mathcal{M}/Q. \tag{5.8}$$

5.3. Let us specialize now to the case where Q is the $\overline{B}$ arising as in Section 4.3 from an effective descent map $p\colon\ E \to B$. The right side of (5.8) becomes $\mathcal{C}^{\overline{B}} \cap \mathcal{M}/\overline{B}$; consider, therefore, which of the maps $f\colon\ A \to B$ are carried into this subcategory by the equivalence $p^\#\colon\ \mathcal{C}/B \to \mathcal{C}^{\overline{B}}$ of (4.10). The image $p^\#(f)$ is the (f_0, f_1, f_2) given by the pullbacks in (4.14), which lies in the pullback-stable $\mathcal{M}$ precisely when f_0 lies in $\mathcal{M}$; and f_0 is another name for the $p^*(f) = g$ of (5.2). Thus $p^\#(f)$ lies in $\mathcal{M}$ precisely when $f \in \mathrm{Spl}(E, p)$. This gives us the *fundamental theorem of Galois Theory*:

THEOREM. *Let $I \dashv H$: $\mathcal{X} \to \mathcal{C}$ be an admissible reflexion for a finitely complete $\mathcal{C}$, and let p: $E \to B$ be an effective descent map in $\mathcal{C}$. Then there is an equivalence of categories*

$$\mathrm{Spl}(E,p) \simeq \mathcal{X}^{I(\overline{B})} \tag{5.9}$$

sending f: $A \to B$ to the image under I of $p^{\#}(f)$.

Since $\overline{B}$ is an equivalence relation and hence a groupoid, the precategory $I(\overline{B})$ in $\mathcal{X}$ is a pregroupoid, by Section 4.1; it is called the *Galois pregroupoid* $\mathrm{Gal}(E,p)$ – or $\mathrm{Gal}_{\mathcal{X}}(E,p)$ when the subcategory $\mathcal{X}$ needs to be specified – of the *extension* (E,p) of the object B of $\mathcal{C}$. Let us record (5.9) again in the form

$$\mathrm{Spl}(E,p) \cong \mathcal{X}^{\mathrm{Gal}(E,p)}. \tag{5.10}$$

Consider in $\mathcal{C}$ the diagram of pullbacks

$$\begin{tikzcd}
E \times_B E \times_B E \times_B E \arrow[r, "m"] \arrow[d, "n"'] & E \times_B E \times_B E \arrow[r, "q"] \arrow[d, "r"] & E \times_B E \arrow[r, "d"] \arrow[d, "c"] & E \arrow[d, "p"] \\
E \times_B E \times_B E \arrow[r, "q"'] & E \times_B E \arrow[r, "d"'] & E \arrow[r, "p"'] & B
\end{tikzcd} \tag{5.11}$$

where

$$d(x,y) = x, \qquad c(x,y) = y, \qquad q(x,y,z) = (x,y), \qquad r(x,y,z) = (y,z),$$
$$m(x,y,z,t) = (x,y,z), \quad \text{and} \quad n(x,y,z,t) = (y,z,t).$$

It is easy to see that, if I preserves the two pullbacks on the left in (5.11), the Galois pregroupoid $\mathrm{Gal}(E,p)$ is actually a category – and hence a groupoid – in $\mathcal{X}$; it is then of course called the *Galois groupoid* of (E,p). When $\mathrm{Gal}(E,p)$ *is* a groupoid, it is a group if and only if IE is the terminal object 1, and is then called the *Galois group* of (E,p); very commonly, an object E of $\mathcal{C}$ with $IE = 1$ is said to be *connected*.

5.4. Let us say a word about the special case where the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ is not only admissible, but has stable units in the sense of Section 3.7, or is even a localization. In the first of these cases, I preserves such pullbacks as (3.11), where $Z \in \mathcal{X}$; in the second, I preserves *all* pullbacks. In the stable-units case, therefore, I preserves *all* the pullback-squares in (5.11) if $B \in \mathcal{X}$; and it does so in the localization case for *all* $B \in \mathcal{C}$. This means that $\mathrm{Gal}(E,p)$ is nothing but the equivalence relation given by the kernel in $\mathcal{X}$ of Ip: $IE \to IB$. If we had a general result that I preserves effective descent maps (which is certainly the case in many important examples), we could now conclude from (5.10) and the equivalence (4.10) that $\mathrm{Spl}(E,p) \simeq \mathcal{X}/IB$; however we can in fact give a direct proof of this:

For any effective descent map $p\colon\ E \to B$, every $f\colon\ A \to B$ in $\mathrm{Spl}(E,p)$ *is a trivial covering in the stable-units case for $B \in \mathcal{X}$, and in the localization case for all B; so that here the equivalence* (5.1) *may be written as $\mathcal{X}/IB \cong$* $\mathrm{Spl}(E,p)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} C \arrow[r, "s"] \arrow[d, "g"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"] & IA \arrow[d, "If"] \\ E \arrow[r, "p"'] & B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{5.12}$$

wherein the left square is a pullback; so that $g \in \mathcal{M}$ since $f \in \mathrm{Spl}(E,p)$. The exterior of (5.12) is equal to the exterior of

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "g"'] & IC \arrow[r, "Is"] \arrow[d, "Ig"] & IA \arrow[d, "If"] \\ E \arrow[r, "\eta_E"'] & IE \arrow[r, "Ip"'] & IB \end{tikzcd} , \tag{5.13}$$

and here the left square is a pullback by Section 3.5 because $g \in \mathcal{M}$, and the right square is a pullback since I preserves the left-hand pullback in (5.12); so the exterior of (5.12) is a pullback. Now the right square of (5.12) is a pullback by Lemma 4.6, giving $f \in \mathcal{M}$.

6. Localization and Stabilization for Factorization Systems

6.1. We turn now to our central results. On the finitely-complete $\mathcal{C}$, let $(\mathcal{E},\mathcal{M})$ be any factorization system; it need not be reflective, although the "Galois theory" case where $(\mathcal{E},\mathcal{M})$ arises as in Section 5.1 from an admissible reflexion is an important one, occurring in our three chief examples below.

A map $f\colon\ A \to B$ is often said to possess a property P *locally* if there is some effective descent map $p\colon\ E \to B$ for which the pullback $p^*(f)$ possesses the property P. Let us write $\mathcal{M}^*$ for the class of those maps that are locally in $\mathcal{M}$; so that $f \in \mathcal{M}^*$ precisely when $p^*(f) \in \mathcal{M}$ for *some* effective descent map $p\colon\ E \to B$. In the Galois-theory case, the maps in $\mathcal{M}^*/B$ are just those we called in Section 5.1 the *coverings* of B, while those in $\mathcal{M}/B$ were called the *trivial coverings* of B. In general, we have of course $\mathcal{M} \subset \mathcal{M}^*$, and every map that is *locally* in $\mathcal{M}^*$ is (by (c) of Section 4.7) already in $\mathcal{M}^*$; we may say that $\mathcal{M}^*$ is formed from $\mathcal{M}$ by *localization*.

This class $\mathcal{M}^*$ is pullback-stable: supposing that $fu = kh$ is a pullback square where $f \in \mathcal{M}^*$, and that $fs = pg$ as in (5.2) is a pullback square with p an effective descent map and with $g \in \mathcal{M}$, form a pullback square $pn = kq$. Now the map $q^*(h) = q^*k^*(f) \cong n^*p^*(f) = n^*(g)$ lies in the pullback-stable $\mathcal{M}$, while q is an effective descent map by (b) of Section 4.7; so that $h \in \mathcal{M}^*$.

We may ask whether $\mathcal{M}^*$ is reflective in $\mathcal{C}^{\mathbf{2}}$ – which by Section 2.12 above is equally to ask that each $\mathcal{M}^*/B$ be reflective in $\mathcal{C}/B$. Counter-examples that we shall give in Section 10 show that it need not be so, even in the Galois-theory situation; yet it is indeed so in many important cases: an article in preparation by Janelidze and Kelly will establish this “reflectivity of coverings” under various alternative and often-satisfied hypotheses.

If we have a case where each $\mathcal{M}^*/B$ *is* reflective in $\mathcal{C}/B$, we may ask whether $\mathcal{M}^*$ is closed under composition; recall from Section 2.12 that this is so precisely when $(\mathcal{M}^{*\uparrow}, \mathcal{M}^*)$ is a factorization system on $\mathcal{C}$. Further counter-examples in Section 10 show that this need not be so; but once again, there are interesting cases where it *is* so.

Recall now, from Section 2.11 above, the class $\mathcal{E}'$ consisting of those maps every pullback of which lies in $\mathcal{E}$; it is the largest pullback-stable class contained in $\mathcal{E}$, and may be said to arise from $\mathcal{E}$ by *stabilization*. It turns out, as we shall see in Proposition 6.7, that $e \downarrow m$ whenever $e \in \mathcal{E}'$ and $m \in \mathcal{M}^*$; so that we always have

$$\mathcal{E}' \subset \mathcal{M}^{*\uparrow}. \tag{6.1}$$

Our counter-examples go on to show – even in the Galois-theory situation – that $(\mathcal{M}^{*\uparrow}, \mathcal{M}^*)$ may well be a factorization system with (6.1) a strict inclusion.

The point of the present article is that, in spite of all these counter-examples, there are important cases – including but not restricted to Galois-theory ones – where $(\mathcal{E}', \mathcal{M}^*)$ is indeed a factorization system on $\mathcal{C}$; in such cases, this stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ may be said to arise from $(\mathcal{E}, \mathcal{M})$ by *simultaneously localizing* $\mathcal{M}$ *and stabilizing* $\mathcal{E}$. (Note that, since $\mathcal{M}^{*\uparrow} \subset \mathcal{M}^{\uparrow} = \mathcal{E}$ because $\mathcal{M}^* \supset \mathcal{M}$, it follows from (6.1) that $\mathcal{M}^{*\uparrow}$ must be $\mathcal{E}'$ whenever $(\mathcal{M}^{*\uparrow}, \mathcal{M}^*)$ is a *stable* factorization system. In fact, by (6.1) and (2.8), $\mathcal{M}^{*\uparrow}$ must be $\mathcal{E}'$ if each $\mathcal{M}^*/B$ is a *reflective* (and not merely a fibrewise-reflective) subfibration of $\mathcal{C}/B$.)

We shall now develop necessary and sufficient conditions on an arbitrary factorization system $(\mathcal{E}, \mathcal{M})$, in order that $(\mathcal{E}', \mathcal{M}^*)$ should be another factorization system; then we apply these to our positive examples in Sections 7, 8, and 9 below, before turning in Section 10 to the counter-examples mentioned above.

We begin with a series of simple observations on relations between $\mathcal{E}$, $\mathcal{M}$, $\mathcal{E}'$, $\mathcal{M}^*$, and effective descent maps.

6.2. Our proof in Section 6.1 above that $\mathcal{M}^*$ is pullback-stable clearly used no property of $\mathcal{M}$ other than its pullback-stability; accordingly we have:

For any pullback-stable class $\mathcal{N}$ of maps in $\mathcal{C}$, the class $\mathcal{N}^*$ given by the maps locally in $\mathcal{N}$ is again pullback-stable.

In particular, the class $\mathcal{E}'^*$ is pullback-stable.

6.3. Although the factorization system $(\mathcal{E}, \mathcal{M})$ that we start with will not in general be stable, it may happen that the $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ of a particular $f\colon A \to B$ has $e \in \mathcal{E}'$; in this case we shall say that *the* $(\mathcal{E}, \mathcal{M})$-*factorization of f is stable*. Using these terms, we can re-phrase an observation from Section 2.11: *if the top and bottom edges of* (2.6) *are* $(\mathcal{E}, \mathcal{M})$-*factorizations, the bottom one of which is stable, and if the exterior of* (2.6) *is a pullback, then each of the interior squares is a pullback.*

6.4. From the factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ we get (just as in the more special case of Section 5.2 above) a factorization system – still called $(\mathcal{E}, \mathcal{M})$ – on $[\mathbf{P}, \mathcal{C}]$, by taking $f\colon P \to Q$ to be in $\mathcal{E}$ or in $\mathcal{M}$ when each of its components $f_i\colon P_i \to Q_i$ is so. Then of course a general $f\colon P \to Q$ has the $(\mathcal{E}, \mathcal{M})$-factorization

$$P \xrightarrow[e]{} R \xrightarrow[m]{} Q, \tag{6.2}$$

where each f_i has the $(\mathcal{E}, \mathcal{M})$-factorization $m_i e_i$. We shall need the following:

If a discrete opfibration $f\colon P \to Q$ has in $[\mathbf{P}, \mathcal{C}]$ the $(\mathcal{E}, \mathcal{M})$-factorization (6.2), *and if the $(\mathcal{E}, \mathcal{M})$-factorization $f_0 = m_0 e_0$ is stable, then e and m are discrete opfibrations.*

Proof. In the commutative diagram

$$\begin{tikzcd}
P_2 \arrow[r, "q"] \arrow[d, "e_2"'] & P_1 \arrow[r, "d"] \arrow[d, "e_1"] & P_0 \arrow[d, "e_0"] \\
R_2 \arrow[r, "q"] \arrow[d, "m_2"'] & R_1 \arrow[r, "d"] \arrow[d, "m_1"] & R_0 \arrow[d, "m_0"] \\
Q_2 \arrow[r, "q"] & Q_1 \arrow[r, "d"] & Q_0
\end{tikzcd},$$

the interior squares on the right are pullbacks by (4.4) and Section 6.3; then e_1, being the pullback of $e_0 \in \mathcal{E}'$, itself lies in $\mathcal{E}'$, so that the interior squares on the left are pullbacks for the same reasons.

Note what are in effect trivial special cases:

6.5. PROPOSITION. *A discrete opfibration $f\colon P \to Q$ lies in $\mathcal{E}$ if $f_0 \in \mathcal{E}'$, and lies in $\mathcal{M}$ if $f_0 \in \mathcal{M}$.*

The following will be central to our argument:

6.6. PROPOSITION. *If $e \in \mathcal{E}'^*$ and $m \in \mathcal{M}^*$ we have $e \downarrow m$.*

Proof. Since $\mathcal{M}^*$ is pullback-stable, it suffices by Proposition 2.6 to exhibit a unique diagonal in any commutative

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "1"] \\ D \arrow[r, "m"'] & B \end{tikzcd} \tag{6.3}$$

with $e \in \mathcal{E}'^*$ and $m \in \mathcal{M}^*$. There are effective descent maps $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ such that $p_1^*(e) \in \mathcal{E}'$ and $p_2^*(m) \in \mathcal{M}$; so that, if $p\colon E \to B$ is the diagonal in the pullback (5.4) of p_1 and p_2, it is by (c) of Section 4.7 an effective descent map with $p^*(e) \in \mathcal{E}'$ and $p^*(m) \in \mathcal{M}$ (since $\mathcal{E}'$ and $\mathcal{M}$ are pullback-stable). Applying to (6.3) the equivalence of categories $p^\#\colon \mathcal{C}/B \to \mathcal{C}^{\overline{B}}$ of (4.10), we get (using the notation of Section 4.4) a commutative square

$$\begin{tikzcd} \bar{A} \arrow[r, "p^\#(e)"] \arrow[d, "p^\#(u)"'] & \overline{B} \arrow[d, "1"] \\ \overline{D} \arrow[r, "p^\#(m)"'] & \overline{B} \end{tikzcd} . \tag{6.4}$$

Recall from that section, and from (4.15) in particular, that $p^\#(e)$ and $p^\#(m)$ are discrete opfibrations with $(p^\#(e))_0 = p^*(e)$ and $(p^\#(m))_0 = p^*(m)$. By Proposition 6.5, we have $p^\#(e) \in \mathcal{E}$ since $(p^\#(e))_0 \in \mathcal{E}'$ and $p^\#(m) \in \mathcal{M}$ since $(p^\#(m))_0 \in \mathcal{M}$; thus there is a unique diagonal in (6.4) and hence – because $p^\#$ is an equivalence – a unique diagonal in (6.3).

We can in fact simplify Proposition 6.6:

6.7. PROPOSITION. *$\mathcal{E}'^* = \mathcal{E}'$; and if $e \in \mathcal{E}'$ and $m \in \mathcal{M}^*$ we have $e \downarrow m$.*

Proof. The second statement being a weakening of Proposition 6.6, we need only the first; and since $\mathcal{E}' \subset \mathcal{E}'^*$ trivially, we need only that $\mathcal{E}'^* \subset \mathcal{E}'$. But since $\mathcal{M} \subset \mathcal{M}^*$, Proposition 6.6 gives $\mathcal{E}'^* \subset \mathcal{M}^\uparrow = \mathcal{E}$; and since $\mathcal{E}'^*$ is pullback-stable by Section 6.2, we in fact have $\mathcal{E}'^* \subset \mathcal{E}'$.

6.8. LEMMA. *If a map $f\colon A \to B$ in $\mathcal{C}$ lies in $\mathcal{M}^*$, and if its $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ is stable, then e is invertible and $f \in \mathcal{M}$.*

Proof. To say that the factorization $f = me$ is stable is to say that $e \in \mathcal{E}'$. Besides having $e \downarrow m$, therefore, we have $e \downarrow f$ by Proposition 6.7; so that e is invertible by the dual of Lemma 2.7.

6.9. We agreed in Section 6.3 to call the $(\mathcal{E}, \mathcal{M})$-factorization $f = me$ of $f\colon A \to B$ *stable* when $e \in \mathcal{E}'$. It is accordingly appropriate to say that the $(\mathcal{E}, \mathcal{M})$-factorization of f is *locally stable* if there is some effective descent map $p\colon E \to B$ for which the $(\mathcal{E}, \mathcal{M})$-factorization of the pullback $p^*(f)$ is stable.

It is clear that each of the classes $\mathcal{E}'$ and $\mathcal{M}^*$ contains the identities and is closed under composition with isomorphisms. It follows therefore from Section 2.8, given Proposition 6.7, that $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every $f\colon A \to B$ has a factorization $f = ng$ with $n \in \mathcal{M}^*$ and $g \in \mathcal{E}'$. Our central result is the following:

THEOREM. *$(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every $(\mathcal{E}, \mathcal{M})$-factorization is locally stable.*

Proof. First suppose that $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system. If $f\colon A \to B$ has the $(\mathcal{E}', \mathcal{M}^*)$-factorization $f = ng$, there is an effective descent map $p\colon E \to B$ for which the pullback $k = p^*(n)$ of the element n of $\mathcal{M}^*$ lies in $\mathcal{M}$. When we now pull back g as well, to get a diagram

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "h"'] & {} \arrow[d, "g"] \\ {} \arrow[r] \arrow[d, "k"'] & {} \arrow[d, "n"] \\ {} \arrow[r, "p"'] & {} \end{tikzcd}$$

in which each square is a pullback, we have $h \in \mathcal{E}'$ since $g \in \mathcal{E}'$. So the $(\mathcal{E}, \mathcal{M})$-factorization kh of $p^*(f)$ is stable.

For the converse, consider any $f\colon A \to B$; by hypothesis, there is some effective descent map $p\colon E \to B$ such that the $(\mathcal{E}, \mathcal{M})$-factorization of $p^*(f)$ is stable. In the notation of Section 4.4, we have the discrete opfibration $p^\#(f)\colon \bar{A} \to \overline{B}$; let its $(\mathcal{E}, \mathcal{M})$-factorization in $[\mathbf{P}, \mathcal{C}]$ be $p^\#(f) = me$. By (4.14), the 0-component of this is nothing but the stable $(\mathcal{E}, \mathcal{M})$-factorization $m_0 e_0$ of $f_0 = p^*(f)$; now it follows from Section 6.4 that m like f is a discrete opfibration. Since $p^\#\colon \mathcal{C}/B \to \mathcal{C}^{\overline{B}}$ is an equivalence of categories, there is some $n\colon C \to B$ in $\mathcal{C}$ with $p^\#(n) \cong m$; and we can take this isomorphism to be an equality $p^\#(n) = m$, after adjusting suitably the factorization me of $p^\#(f)$. Then again, since $p^\#$ is an equivalence, there is a unique $g\colon f \to n$ in $\mathcal{C}/B$ with $p^\#(g) = e$. Besides the pullback square (4.13) for f, we have a similar one for n and one like (4.17) for g; the 0-components of these are the pullback squares in

$$\begin{tikzcd} A_0 \arrow[r, "s"] \arrow[d, "e_0"'] & A \arrow[d, "g"] \\ C_0 \arrow[r, "t"'] \arrow[d, "m_0"'] & C \arrow[d, "n"] \\ B_0 \arrow[r, "p"'] & B \end{tikzcd} .$$

Since p is an effective descent map and $m_0 \in \mathcal{M}$, we have $n \in \mathcal{M}^*$; again, since t too is an effective descent map by (b) of Section 4.7 and since $e_0 \in \mathcal{E}'$, we have $g \in \mathcal{E}'^*$, and so $g \in \mathcal{E}'$ by Proposition 6.7. Thus we have as desired an $(\mathcal{E}', \mathcal{M}^*)$-factorization ng of f.

6.10. The theorem above may not be easy to apply: where is one to search for an effective descent map $p\colon E \to B$ for which the $(\mathcal{E}, \mathcal{M})$-factorization of $p^*(f)$ is stable? The following notion – which leads only to a *sufficient* condition – narrows the search in suitable cases. Call an object E of $\mathcal{C}$ *stabilizing* if the $(\mathcal{E}, \mathcal{M})$-factorization of every map $x\colon D \to E$ is stable. The theorem above clearly gives:

PROPOSITION. $(\mathcal{E}', \mathcal{M}^*)$ *is a factorization system if, for each* $B \in \mathcal{C}$, *there is an effective descent map* $p\colon E \to B$ *with* E *stabilizing; in which case we say that* there are enough stabilizing objects.

6.11. Let us call an object E of $\mathcal{C}$ *projective*, in our present context, if $\mathcal{C}(E, p)$ is a surjection in **Set** for each effective descent map p; and let us say that *there are* *enough projectives* if for every B there is some effective descent map $p\colon\ E \to B$ with E projective. The following gives, in a sense, a necessary and sufficient condition; although it is clearly no easier to apply than the criterion of Proposition 6.10 above:

PROPOSITION. *When there are enough projectives, $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system if and only if every projective is stabilizing.*

Proof. The "if" part is clear from Proposition 6.10. For the "only if" part, let $f\colon\ A \to B$ with B projective; by Theorem 6.9, there is an effective descent map $p\colon\ E \to B$ for which the $(\mathcal{E}, \mathcal{M})$-factorization of $p^*(f)$ is stable; because B is projective, there is some $h\colon\ B \to E$ with $ph = 1$; now the $(\mathcal{E}, \mathcal{M})$-factorization of $h^*p^*(f)$, like that of $p^*(f)$, is stable; but $h^*p^*(f)$, to within isomorphism, is f.

6.12. REMARK. Note finally that the question whether $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system becomes trivial when the original factorization system $(\mathcal{E}, \mathcal{M})$ is stable, so that $\mathcal{E}' = \mathcal{E}$; for then *every* object is stabilizing. It follows that in this case we have $\mathcal{M}^* = \mathcal{M}$ (which it is also quite easy to prove directly).

7. The (Monotone, Light)-Factorization for Compact Spaces

7.1. For our first example we take $\mathcal{C}$ to be the category of compact Hausdorff spaces. In this category the monomorphisms are the injections and every monomorphism is regular, while the epimorphisms are the surjections and every epimorphism is regular; moreover $\mathcal{C}$, being monadic over **Set** by a result of Manes, is an exact category.

Let $\mathcal{X}$ be the full subcategory of $\mathcal{C}$ given by the totally disconnected spaces (often called the Stone spaces); it is of course closed in $\mathcal{C}$ under both subobjects and quotient objects. It is well known that the inclusion H of $\mathcal{X}$ into $\mathcal{C}$ has a left adjoint I, the unit $\eta_A\colon\ A \to IA$ for which is the canonical projection of A onto the set IA of its components, this set being given the quotient topology with respect to η_A; see, for example, [2, Ch. II, §4, No. 4, Prop. 7].

We take for $(\mathcal{E}, \mathcal{M})$ the reflective factorization system $\Phi\mathcal{X}$ derived as in Section 3.1 above from this reflexion. By (3.2), a map $f\colon\ A \to B$ lies in $\mathcal{E}$ precisely when f induces a bijection between the components of A and those of B. If f lies in $\mathcal{E}'$ – that is, if every pullback of f lies in $\mathcal{E}$ – then in particular every pullback of f along $b\colon\ 1 \to B$ lies in $\mathcal{E}$; which is to say that the fibre $f^{-1}(b)$ is connected. Following Whyburn [25], we call a map f *monotone* when each of its fibres is connected; we remind the reader that a connected space has *one* component, and is accordingly never empty. Observe now that every pullback of a monotone map is monotone; and that any monotone f lies in $\mathcal{E}$, since the inverse image of a component H of B must be a single component of

A. (For if $f^{-1}(H)$ is the disjoint union of two non-empty closed sets K and L, each $f^{-1}(b)$ for $b \in H$ lies wholly in K or in L, so that H is the disjoint union of the non-empty closed sets $f(K)$ and $f(L)$ – a contradiction.) Thus

$$\mathcal{E}' \textit{ consists of the monotone maps.} \tag{7.1}$$

7.2. Since the unit $\eta_B\colon\ B \to IB$ is clearly monotone, it lies in $\mathcal{E}'$, and every pullback of it lies in $\mathcal{E}$; thus the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ has stable units in the sense of Section 3.7, and is *a fortiori* admissible in the sense of Section 3.6; so that we find ourselves in the Galois-theory situation of Section 5.1. By Sections 3.6 and 3.5, the map $f\colon\ A \to B$ lies in $\mathcal{M}$ – or equivalently, is a *trivial covering* in the sense of Section 5.1 – precisely when (3.9) is a pullback; to require this is clearly to require that, for each component H of B, every component of $f^{-1}(H)$ is mapped by f bijectively – and hence homeomorphically – onto H. Note that the reflexion is not left exact: I does not preserve the equalizer of the two maps from 1 to the unit interval picking out its two ends.

The Stone–Čech compactification βS of any discrete space S, being a subspace of some 2^Y, is totally disconnected; that is, $\beta S \in \mathcal{X}$. Any map $f\colon\ A \to \beta S$ factorizes therefore as

$$A \xrightarrow[\eta_A]{} IA \xrightarrow[m]{} \beta S\ ; \tag{7.2}$$

and here m, being a map in $\mathcal{X}$, lies in $\mathcal{M}$. Because $\eta_A \in \mathcal{E}'$, we conclude that the object βS is *stabilizing* in the sense of Section 6.10. Now in fact there are *enough* stabilizing objects. For, the category $\mathcal{C}$ being exact, the effective descent maps (by (a) of Section 4.7) are the regular epimorphisms, which here are the surjections; and for each $B \in \mathcal{C}$ we have the canonical surjection $\beta|B| \to B$, where $|B|$ is the underlying set of B. By Proposition 6.10, therefore, we conclude that $(\mathcal{E}', \mathcal{M}^*)$ *is a factorization system*.

It remains to determine the class $\mathcal{M}^*$. The map $f\colon\ A \to B$ in $\mathcal{C}$ is said – see for example Whyburn [25] – to be *light* if each of its fibres is totally disconnected. The fact that every map $f\colon\ A \to B$ of compact metric spaces has a factorization $f = ng$ with n light and g monotone was first proved by Eilenberg [7]; by establishing the following, we recapture this classical result, without the "metric" restriction.

7.3. THEOREM. *The classes $\mathcal{E}'$ and $\mathcal{M}^*$ above consist respectively of the monotone maps and the light ones, and constitute a stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ on the category of compact Hausdorff spaces.*

Proof. It remains only to identify $\mathcal{M}^*$ with the light maps. If $f\colon\ A \to B$ lies in $\mathcal{M}^*$, so too by Section 6.2 does any pullback of f; in particular the pullback along $b\colon\ 1 \to B$. Thus, for some surjection $E \to 1$, the pullback of f along

$$E \longrightarrow 1 \xrightarrow[b]{} B$$

lies in $\mathcal{M}$. But then, for $e \in \mathcal{E}$, the pullback of f along

$$1 \xrightarrow[e]{} E \longrightarrow 1 \xrightarrow[b]{} B$$

lies in $\mathcal{M}$. This pullback being $f^{-1}(b) \to 1$, we conclude (either from Section 3.4, or from the explicit description of $\mathcal{M}$ above) that $f^{-1}(b)$ is totally disconnected. Thus f is light.

Conversely, if $f\colon\ A \to B$ is light, so too is its pullback along the canonical surjection $\beta|B| \to B$; this pullback, since it has totally disconnected fibres and a totally disconnected codomain $\beta|B|$, is a trivial covering by the first paragraph of Section 7.2; thus it lies in $\mathcal{M}$, and f lies in $\mathcal{M}^*$.

7.4. Observe that $\mathcal{M}^*/1 = \mathcal{X}$; since $\mathcal{M}^* \neq \mathcal{M}$, it follows that the factorization system $(\mathcal{E}', \mathcal{M}^*)$ is not reflective.

7.5. Once we know the existence of a factorization $f = ng$ with n light and g monotone, we can describe it explicitly; for the codomain C of g must be the set of all the components of all the fibres of f, with the quotient topology corresponding to the evident map g.

7.6. As an illustration of the fact that we can apply Theorem 6.9, in the form of Proposition 6.10, to a factorization system that is not reflective, let $\mathcal{C}, \mathcal{X}, \mathcal{E}$ and $\mathcal{M}$ be as above, but take as a new starting-point the non-reflective factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ of (3.12), where $\mathcal{F}$ consists of the surjections in $\mathcal{C}$ (the epimorphisms) and $\mathcal{N}$ of the injections (the monomorphisms). So $\overline{\mathcal{E}}$ consists of the *surjections* $f\colon\ A \to B$ which induce a bijection between the components of A and those of B – a class that is strictly smaller than $\mathcal{E}$. In fact, however, $\overline{\mathcal{E}}'$ and $\mathcal{E}'$ coincide: for every $f \in \mathcal{E}'$ is surjective, its fibres being connected and thus non-empty. Since the factorization (7.2) of any $A \to \beta S$ has $m \in \mathcal{M} \subset \overline{\mathcal{M}}$ and $\eta_A \in \mathcal{E}' = \overline{\mathcal{E}}'$, it follows that βS is stabilizing not only for $(\mathcal{E}, \mathcal{M})$ but also for $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$. By Proposition 6.10, therefore, $(\overline{\mathcal{E}}', \overline{\mathcal{M}}^*)$ is another stable factorization system. However, since $\overline{\mathcal{E}}'$ coincides with $\mathcal{E}'$, it must be the case that $\overline{\mathcal{M}}^*$ coincides with $\overline{\mathcal{M}}$ – as the reader may easily verify directly, with the knowledge from Section 3.9 that $f\colon\ A \to B$ lies in $\overline{\mathcal{M}}$ if and only if $\langle f, \eta_A \rangle$ is injective, which is just to say that, for each component H of B, every component of $f^{-1}(H)$ is mapped by f *injectively* into H.

7.7. Note that this reflexion of $\mathcal{C}$ onto $\mathcal{X}$, being admissible, provides a new application for the Galois theory of Section 5 – namely to the classification of the light maps $f\colon\ A \to B$ in $\mathcal{C}$, these being the “coverings” in this case. By Section 5.1 we have in fact Cov $B \simeq \mathrm{Spl}(\beta|B|, p)$, where $p\colon\ \beta|B| \to B$ is the canonical projection; for $\beta|B|$ is projective in $\mathcal{C}$. Thus (5.10) gives

$$\mathrm{Cov} B \simeq \mathcal{X}^{\mathrm{Gal}(\beta|B|, p)}. \tag{7.3}$$

Since in the present case the reflexion has stable units, Section 5.4 tells us that every covering of B is trivial when $B \in \mathcal{X}$; but this is of course clear directly, and we in fact used it in the proof of Theorem 7.3.

From another point of view, what we have here is a Galois theory for the category of C^*-algebras; for this of course is the dual of $\mathcal{C}$.

8. The Example of Hereditary Torsion Theories

8.1. Our second example is in fact a family of examples, the central and paradigmatic one being that where $\mathcal{C}$ is the category of abelian groups and $(\mathcal{E}, \mathcal{M})$ is the reflective factorization system $\Phi\mathcal{X}$ corresponding to the reflective full subcategory $\mathcal{X}$ of torsion-free groups. The general example in the family involves the abstraction of this situation provided by the notion, recalled below, of an *hereditary torsion theory* on an abelian category $\mathcal{C}$ (which is in fact coextensive with that of an *hereditary radical*).

The present example, although on the face of things concerned with quite a different area of mathematics, in fact behaves formally very much indeed like the last one – roughly with "torsion object" replacing "connected space" and "torsion-free object" replacing "totally disconnected space". As in that example, the factorization system $(\mathcal{E}, \mathcal{M})$ we begin with arises as $\Phi\mathcal{X}$ from an admissible reflexion – indeed one with stable units – of $\mathcal{C}$ onto a full subcategory $\mathcal{X}$; so that $\mathcal{M}^*$ is the class of coverings classified by the corresponding Galois theory. One difference is that, this time, the application of Galois theory fails to be a *new* one – but only just: an article [16], very recently prepared by Janelidze, Márki, and Tholen, considers a generalization of Janelidze's Galois theory to the situation of a general *radical*, and so contains the Galois theory of the present example as a special case. The new insights that *are* provided by our results here are that the class $\mathcal{M}^*$ of coverings is part of a factorization system $(\mathcal{M}^{*\uparrow}, \mathcal{M}^*)$, for which in fact $\mathcal{M}^{*\uparrow} = \mathcal{E}'$. Indeed – and this, by Section 7.6, is in stark contrast to the situation in the last example – if we take for $\overline{\mathcal{E}}$ the class of *epimorphisms* in $\mathcal{E}$, then the factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ of Section 3.9 *coincides* with $(\mathcal{E}', \mathcal{M}^*)$.

Supposing for the remainder of this section that $\mathcal{C}$ is abelian, we begin by recalling the notions of torsion theory and of hereditary torsion theory for such a $\mathcal{C}$, examining them in the light of the results of Section 3 above. (A more complete analysis of them in this light can be found in [6], along with generalizations to various non-abelian $\mathcal{C}$.)

8.2. We may identify a full subcategory $\mathcal{Y}$ of the abelian $\mathcal{C}$ with the set of its objects. Given such a $\mathcal{Y}$ we define a new full subcategory $\mathcal{Y}^{\rightarrow}$ to consist of those X for which every map $f\colon\ Y \to X$ with $Y \in \mathcal{Y}$ is zero. Clearly $\mathcal{Y}^{\rightarrow}$ is closed in $\mathcal{C}$ under subobjects; when it is reflective, therefore, the units of the reflexion are epimorphic. (In fact – see [6, Thm. 5.7] – it always *is* reflective if $\mathcal{C}$ admits all cointersections of quotient objects.) Again, it is immediate that $\mathcal{Y}^{\rightarrow}$ is *closed* *in $\mathcal{C}$ under extensions* – in the sense that, if a subobject A of B lies in $\mathcal{Y}^{\rightarrow}$ and the quotient B/A lies in $\mathcal{Y}^{\rightarrow}$, then B itself lies in $\mathcal{Y}^{\rightarrow}$.

Dually we define, for a full subcategory $\mathcal{X}$, the full subcategory $\mathcal{X}^{\leftarrow}$; of course we always have $\mathcal{X}^{\leftarrow\rightarrow\leftarrow} = \mathcal{X}^{\leftarrow}$. A pair $(\mathcal{Y}, \mathcal{X})$ of full subcategories for which $\mathcal{Y}^{\rightarrow} = \mathcal{X}$ and $\mathcal{X}^{\leftarrow} = \mathcal{Y}$ is called a *torsion theory* on $\mathcal{C}$, provided that $\mathcal{X}$ is reflective and $\mathcal{Y}$ coreflective in $\mathcal{C}$. To obtain more insight into torsion theories, we begin by examining $\mathcal{X}^{\leftarrow}$ for a general reflective $\mathcal{X}$.

8.3. Suppose, then, that η_A: $A \to IA$ is a reflexion of $\mathcal{C}$ onto $\mathcal{X}$; write $(\mathcal{E}, \mathcal{M})$ for the prefactorization system $\Phi\mathcal{X}$ of Section 3.1, and denote the kernel of η_A by ζ_A: $TA \to A$.

PROPOSITION. *The following conditions on $A \in \mathcal{C}$ are equivalent:*

(i) $A \in \mathcal{X}^{\leftarrow}$*;*
(ii) $\eta_A = 0$*;*
(iii) ζ_A: $TA \to A$ *is invertible;*
(iv) $IA = 0$*;*
(v) *the map* $0 \to A$ *lies in* $\mathcal{E}$.

Proof. Clearly (i), (ii), and (iii) are equivalent, while (iv) and (v) are equivalent by (3.2). The equivalence of (ii) and (iv) is a simple property of any reflexion in a pointed category.

8.4. As we remarked in Section 8.2, $\mathcal{X}^{\leftarrow}$ is coreflective if $\mathcal{C}$ admits all intersections of subobjects; but, whether the latter is true or not, $\mathcal{X}^{\leftarrow}$ is coreflective by (v) of Proposition 8.3 whenever $\mathcal{X}$ is reflective and the prefactorization system $(\mathcal{E}, \mathcal{M})$ is a factorization system: for, dualizing observations in Section 3.4, we find the coreflexion of A into $\mathcal{X}^{\leftarrow}$ by taking the $(\mathcal{E}, \mathcal{M})$-factorization of $0 \to A$. In particular, $\mathcal{X}^{\leftarrow}$ is coreflective whenever the reflexion I of $\mathcal{C}$ onto $\mathcal{X}$ is *simple* in the sense of Section 3.5. When $\mathcal{X}^{\leftarrow}$ *is* coreflective, the coreflexion σ_A: $SA \to A$ is monomorphic, since $\mathcal{X}^{\leftarrow}$ is closed in $\mathcal{C}$ under quotient objects. Because $SA \in \mathcal{X}^{\leftarrow}$ and $IA \in \mathcal{X}$, we have

$$\eta_A \sigma_A = 0\colon\ SA \to IA. \tag{8.1}$$

Accordingly σ_A factorizes through ζ_A via a monomorphism

$$\psi_A\colon\ SA \to TA. \tag{8.2}$$

PROPOSITION. *For a reflective $\mathcal{X}$ the following are equivalent:*

(i) $\mathcal{X}^{\leftarrow}$ *is coreflective and* ψ: $S \to T$ *is invertible;*
(ii) $TA \in \mathcal{X}^{\leftarrow}$ *for each* A*;*
(iii) $IT = 0$*;*
(iv) ζT: $T^2 \to T$ *is invertible.*

Proof. (ii), (iii), and (iv) are equivalent by Proposition 8.3. Since the coreflexion S has $\sigma S\colon\ S^2 \cong S$, we conclude that (i) implies (iv). It remains to show that (ii) implies (i); that is, that $\zeta_A\colon\ TA \to A$ is a coreflexion of A into $\mathcal{X}^{\leftarrow}$ when (ii) holds. Indeed, let $f\colon\ Y \to A$ with $Y \in \mathcal{X}^{\leftarrow}$; then $\eta_A f = 0$, so that $f = \zeta_A g$ for some g – a unique one, since ζ_A is monomorphic.

8.5. Following [6], we may call the reflexion I of $\mathcal{C}$ onto $\mathcal{X}$ *normal* if it satisfies the equivalent conditions of Proposition 8.4. We shall be concerned only with the case where $\mathcal{X}$ is closed under subobjects, so that each η_A is epimorphic; then the first assertion of the following gives a simple characterization of normality:

PROPOSITION. *When the reflexion $\eta_A\colon\ A \to IA$ of $\mathcal{C}$ onto $\mathcal{X}$ has every η_A epimorphic, it is normal if and only if it has stable units. The reflexion is certainly normal if $\mathcal{X}$ is closed in $\mathcal{C}$, not only under subobjects, but also under extensions.*

Proof. Suppose that I is normal, let the square in the diagram

$$\begin{tikzcd} & C \arrow[r, "u"] \arrow[d, "v"] & A \arrow[d, "g"] \\ TB \arrow[ur, "k"] \arrow[r, "\zeta_B"'] & B \arrow[r, "\eta_B"'] & IB \end{tikzcd} \tag{8.3}$$

be a pullback, and define k by $uk = 0, vk = \zeta_B$; then, by a classical argument, $k = \ker u$. Since η_B is epimorphic, so is its pullback u; whence $u = \operatorname{coker} k$. Because I is a left adjoint, we have $Iu = \operatorname{coker}(Ik)$ in $\mathcal{X}$. But $ITB = 0$ by the normality of I, so that $Ik = 0$ and Iu is invertible; thus $u \in \mathcal{E}$, as required. For the converse, consider (8.3) in the case $A = 0$; then $C = TB$ and $v = \zeta_B$. Since the reflexion has stable units, Iu is invertible, so that $ITB = IC \cong IA = 0$; hence I is normal.

Suppose now that $\mathcal{X}$ is also closed in $\mathcal{C}$ under extensions, and consider the pushout

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[d, "\zeta_A"'] & ITA \arrow[d, "v"] \\ A \arrow[r, "u"'] & X \end{tikzcd} , \tag{8.4}$$

noting that u like η_{TA} is epimorphic, while v like ζ_A is monomorphic. Since $\eta_A\zeta_A = 0$, there is a unique $w\colon\ X \to IA$ defined by $wu = \eta_A$, $wv = 0$; and $w =$

coker v by the dual of the classical argument mentioned above. Because ITA and IA belong to $\mathcal{X}$, which is closed under extensions, we have $X \in \mathcal{X}$; accordingly $u = t\eta_A$ for some t, which like u is epimorphic. Now $\eta_A = wu = wt\eta_A$ gives $wt = 1$, whence w is invertible since t is epimorphic. Thus $v = 0$, so that $ITA = 0$ and I is normal.

8.6. To give a torsion theory $(\mathcal{Y}, \mathcal{X})$, therefore, it suffices to give a reflective full subcategory $\mathcal{X}$ closed under subobjects and extensions, and to set $\mathcal{Y} = \mathcal{X}^{\leftarrow}$. For then the reflexion I of $\mathcal{C}$ onto $\mathcal{X}$ is normal by Proposition 8.5, whence $\mathcal{Y}$ is coreflective by Proposition 8.4, the coreflexion $\sigma_A\colon\ SA \to A$ being the kernel of $\eta_A\colon\ A \to IA$, so that we have an exact sequence

$$0 \longrightarrow SA \xrightarrow[\sigma_A]{} A \xrightarrow[\eta_A]{} IA \longrightarrow 0\ ; \tag{8.5}$$

and now, by Proposition 8.3, we have $A \in \mathcal{Y}^{\rightarrow}$ precisely when $\sigma_A = 0$; which by (8.5) is to say that $\mathcal{Y}^{\rightarrow} = \mathcal{X}$.

The paradigmatic example of a torsion theory is that where $\mathcal{C}$ is the category of abelian groups and $\mathcal{X}$ the reflective subcategory of torsion-free groups; then SA is the torsion subgroup of A, and $\mathcal{Y}$ consists of the torsion groups. By analogy with this case, the objects of $\mathcal{Y}$ in a general torsion theory may be called *torsion objects*, and those of $\mathcal{X}$ *torsion-free objects*.

8.7. Since the reflexion I onto $\mathcal{X}$ in the case of a torsion theory has stable units by Proposition 8.5, it is *a fortiori* simple by Section 3.7, so that the corresponding prefactorization system $\Phi\mathcal{X} = (\mathcal{E}, \mathcal{M})$ is a factorization system by Section 3.5. The following provides an elegant alternative description of torsion theories, in terms of factorization systems:

PROPOSITION. *If $(\mathcal{Y}, \mathcal{X})$ is a torsion theory, the coreflective factorization system corresponding to the coreflective $\mathcal{Y}$ coincides with the reflective factorization system $(\mathcal{E}, \mathcal{M})$ corresponding to the reflective $\mathcal{X}$. Conversely, if a factorization system $(\mathcal{E}, \mathcal{M})$ is both reflective and coreflective, we have a torsion theory $(0/\mathcal{E}, \mathcal{M}/0)$.*

Proof. For $f\colon\ A \to B$ we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & SA \arrow[r, "\sigma_A"] \arrow[d, "Sf"'] & A \arrow[r, "\eta_A"] \arrow[d, "f"'] & IA \arrow[r] \arrow[d, "If"] & 0 \\
0 \arrow[r] & SB \arrow[r, "\sigma_B"'] & B \arrow[r, "\eta_B"'] & IB \arrow[r] & 0
\end{tikzcd} \tag{8.6}$$

Since the reflexion I is simple, we have by Section 3.5 that $f \in \mathcal{M}$ if and only if the right square in (8.6) is a pullback; in an abelian $\mathcal{C}$, however, this is equally to say that Sf is invertible, or that f belongs to the class $\widetilde{\mathcal{M}}$ of the coreflective factorization system $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{M}})$ corresponding to the coreflective $\mathcal{Y}$.

Suppose conversely that the factorization system $(\mathcal{E}, \mathcal{M})$ is both reflective and coreflective, and write $\mathcal{X}$ for the reflective $\mathcal{M}/0$ and $\mathcal{Y}$ for the coreflective $0/\mathcal{E}$. Then $\mathcal{Y} = \mathcal{X}^{\leftarrow}$ by Proposition 8.3, and dually $\mathcal{X} = \mathcal{Y}^{\rightarrow}$.

8.8. A torsion theory $(\mathcal{Y}, \mathcal{X})$ is said to be *hereditary* if $\mathcal{Y}$ is closed in $\mathcal{C}$ under subobjects. Clearly the basic torsion theory (torsion groups, torsion-free groups) on **Ab** is hereditary, but its dual ((torsion-free groups)$^{\mathrm{op}}$, (torsion groups)$^{\mathrm{op}}$) on **Ab**$^{\mathrm{op}}$ is not. We need just one fact about hereditary torsion theories:

PROPOSITION. *The torsion theory $(\mathcal{Y}, \mathcal{X})$ is hereditary if and only if the coreflexion S of $\mathcal{C}$ onto $\mathcal{Y}$ is left exact as a functor $S\colon \mathcal{C} \to \mathcal{C}$.*

Proof. If $\mathcal{Y}$ is closed under subobjects, the kernel in $\mathcal{C}$ of a map in $\mathcal{Y}$ lies in $\mathcal{Y}$, and is therefore the kernel in $\mathcal{Y}$; thus the inclusion $\mathcal{Y} \to \mathcal{C}$ preserves kernels, whence $S\colon \mathcal{C} \to \mathcal{C}$ preserves kernels since the coreflexion $\mathcal{C} \to \mathcal{Y}$, being a right adjoint, does so. Since S automatically preserves finite products when $\mathcal{C}$ is abelian, S is left exact. For the converse, consider a monomorphism $i\colon A \to Y$ with $Y \in \mathcal{Y}$, and let i be the kernel in $\mathcal{C}$ of $f\colon Y \to B$. Since S is left exact, Si is the kernel of Sf. Because σ_Y is invertible, there is some $j\colon A \to SY$ with $\sigma_Y j = i$. Since $\sigma_B.Sf.j = f\sigma_Y j = fi = 0$ and σ_B is monomorphic, we have $Sf.j = 0$. Accordingly $j = Si.k$ for some $k\colon A \to SA$, giving $i = \sigma_Y j = \sigma_Y.Si.k = i\sigma_A k$ and hence $1 = \sigma_A k$. Since σ_A is monomorphic, it is invertible; so that $A \in \mathcal{Y}$, as desired.

8.9. REMARK. A subfunctor $\sigma\colon S \to 1$ of the identity is called an *hereditary radical* if S is left exact (which implies that $S^2 \cong S$) and if the cokernel $\eta\colon 1 \to I$ of σ has $I^2 \cong I$. It follows at once that hereditary radicals on $\mathcal{C}$ are in bijection with hereditary torsion theories.

When injective envelopes exist in $\mathcal{C}$, it is classical that there is a bijection between hereditary torsion theories $(\mathcal{Y}, \mathcal{X})$ on $\mathcal{C}$ and localizations $\mathcal{Z}$ of $\mathcal{C}$, where $\mathcal{X}$ is found as the subobject-closure of $\mathcal{Z}$. For an account in the present language, see [6, Thm. 9.17].

8.10. Suppose henceforth that $(\mathcal{E}, \mathcal{M})$ is the reflective factorization system $\Phi\mathcal{X}$ where $(\mathcal{Y}, \mathcal{X})$ is a torsion theory on $\mathcal{C}$. For a map $f\colon A \to B$ we have by Proposition 8.7

$$f \in \mathcal{M} \quad \text{if and only if} \quad Sf \text{ is invertible.} \tag{8.7}$$

The class $\mathcal{E}$ consists of those f for which If is invertible; these are the f for which

$$\text{(i)} \quad f^*(SB) \leqslant SA, \qquad \text{(ii)} \quad \mathrm{im} f + SB = B; \tag{8.8}$$

for (i) is the assertion that If is monomorphic in (8.6) and (ii) the assertion that it is epimorphic. Since $SA \leqslant f^*(SB)$ for any f, we may equally write (i) as an equality

$$f^*(SB) = SA; \tag{8.9}$$

which is the assertion that the left square in (8.6) is a pullback.

Write $\overline{\mathcal{E}} \subset \mathcal{E}$ for the class of epimorphisms in $\mathcal{E}$; by (8.8), it consists of the epimorphisms f satisfying (8.9). As in Section 3.9, we have a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ where $f \in \overline{\mathcal{M}}$ if and only if $\langle f, \eta_A \rangle \colon A \to B \times IA$ is monomorphic; that is, if and only if $\ker f \cap SA = 0$. Thus

$$f \in \overline{\mathcal{M}} \quad \text{if and only if} \quad \ker f \text{ is torsion-free.} \tag{8.10}$$

This being so, it is easy to give explicitly the $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$-factorization $f = me$ of $f\colon A \to B$; one finds that e is the canonical epimorphism $A \to A/K$, where $K = SA \cap \ker f$.

8.11. PROPOSITION. *The class $\overline{\mathcal{E}}$ is pullback-stable if the torsion theory $(\mathcal{Y}, \mathcal{X})$ is hereditary; and then $\overline{\mathcal{E}} \subset \mathcal{E}'$.*

Proof. Let g be a pullback $h^*(f)$ of $f \in \overline{\mathcal{E}}$; then g like f is epimorphic, so that it remains to establish for g the analogue of (8.9). In the diagram

$$\begin{tikzcd}
SC \arrow[r, "Sk"] \arrow[d, "Sg"'] & SA \arrow[r, "\sigma_A"] \arrow[d, "Sf"] & A \arrow[d, "f"] \\
SD \arrow[r, "Sh"'] & SB \arrow[r, "\sigma_B"'] & B
\end{tikzcd},$$

the left square is a pullback because the left-exact S preserves pullbacks, while the right square is a pullback by (8.9); hence the exterior is a pullback. This exterior, however, is equally the exterior of

$$\begin{tikzcd}
SC \arrow[r, "\sigma_C"] \arrow[d, "Sg"'] & C \arrow[r, "k"] \arrow[d, "g"] & A \arrow[d, "f"] \\
SD \arrow[r, "\sigma_D"'] & D \arrow[r, "h"'] & B
\end{tikzcd},$$

whose right square is a pullback; so the left square too is a pullback, as desired. Then, since the stable $\overline{\mathcal{E}}$ is contained in $\mathcal{E}$, it is contained in $\mathcal{E}'$.

8.12. PROPOSITION. *When the torsion theory $(\mathcal{Y}, \mathcal{X})$ is hereditary, every object X of $\mathcal{X}$ is stablizing for $(\mathcal{E}, \mathcal{M})$.*

Proof. A map $f\colon A \to X$ with $X \in \mathcal{X}$ factorizes as $m\eta_A$ for some $m\colon IA \to X$. Since $\eta_A \in \mathcal{E}$ by (3.4) while m, being a map in $\mathcal{X}$, lies in $\mathcal{M}$, this is the $(\mathcal{E}, \mathcal{M})$-factorization of f. But η_A, being epimorphic, in fact lies in $\overline{\mathcal{E}}$; by Proposition 8.11, therefore, it lies in $\mathcal{E}'$, and is thus stabilizing in the sense of Section 6.10.

8.13. THEOREM. *Let $(\mathcal{Y}, \mathcal{X})$ be an hereditary torsion theory, and suppose that for each $A \in \mathcal{C}$ there is an epimorphism $p\colon X \to A$ with $X \in \mathcal{X}$. Then $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system, coinciding with $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$.*

Proof. Since, by (a) of Section 4.7, the effective descent maps in the abelian and hence exact $\mathcal{C}$ are the epimorphisms, $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system by Proposition 6.10. It remains to show that $\overline{\mathcal{E}} = \mathcal{E}'$; by Proposition 8.11 it suffices for this to show that $\mathcal{E}' \subset \overline{\mathcal{E}}$, or equally that every $g\colon C \to D$ in $\mathcal{E}'$ is epimorphic. Choose an epimorphism $p\colon B \to D$ with $B \in \mathcal{X}$, and form the pullback

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "p"'] & D \end{tikzcd} .$$

Since p is epimorphic, g will be epimorphic if f is so – which is indeed the case, for im $f + SB = B$ by (8.8) because $f \in \mathcal{E}$, while $SB = 0$ because $B \in \mathcal{X}$.

8.14. REMARK. If there are enough projectives in $\mathcal{C}$, the condition that every $A \in \mathcal{C}$ be an epimorphic image of some $X \in \mathcal{X}$ is equivalent to the condition that all the projectives lie in $\mathcal{X}$ (which we take to be replete). For the non-trivial direction, consider a projective P, and let $p\colon X \to P$ be an epimorphism with $X \in \mathcal{X}$; then, since P is projective, we have $1_P = ph$ for some $h\colon P \to X$; whence P, as a retract of the object X of the reflective $\mathcal{X}$, itself lies in $\mathcal{X}$.

8.15. REMARK. Our paradigmatic hereditary torsion theory on **Ab** certainly has this property that the projectives are torsion-free, and hence provides an example of Theorem 8.13; as do other hereditary torsion theories on **Ab**, such as that where $\mathcal{Y}$ consists of the 2-torsion groups and $\mathcal{X}$ of the groups without 2-torsion. There are many further such examples where $\mathcal{C}$ is a module-category R-Mod for some ring R.

Yet there are hereditary torsion theories on a category $\mathcal{C}$ of the form R-Mod for which the projective R does not lie in $\mathcal{X}$. Take, for instance, $\mathcal{C}$ to be **Ab** $\times$ **Ab** =

$(\mathbf{Z} \times \mathbf{Z})$-Mod, with $\mathcal{X}$ given by the objects $(A, 0)$ and $\mathcal{Y}$ by the objects $(0, B)$; certainly $R = \mathbf{Z} \times \mathbf{Z} = (\mathbf{Z}, \mathbf{Z})$ does not lie in $\mathcal{X}$. However it is quite trivial that $(\mathcal{E}', \mathcal{M}^*)$ *is* a factorization system in this case; it coincides indeed with $(\mathcal{E}, \mathcal{M})$. For lack of time, we have not determined whether *every* hereditary torsion theory, say on $\mathcal{C} = R$-Mod, gives rise to a factorization system $(\mathcal{E}', \mathcal{M}^*)$.

9. The (Separable, Purely Inseparable) Factorization System for Algebras

9.1. Turning to yet another seemingly diverse area of mathematics, we consider for a field k the category $\mathcal{P}$ of finite-dimensional commutative k-algebras, and the full subcategory $\mathcal{Q}$ of $\mathcal{P}$ given by the semi-simple algebras; we shall take for our $\mathcal{C}$ either the dual $\mathcal{P}^{\mathrm{op}}$ of $\mathcal{P}$, or else the dual $\mathcal{Q}^{\mathrm{op}}$ of $\mathcal{Q}$. In fact we deal primarily with the latter case, since this suffices to illustrate the main points while admitting a significantly shorter treatment; we describe the results for the former case in a remark, leaving the reader to verify the details.

Each of $\mathcal{P}$ and $\mathcal{Q}$ admits finite limits, preserved by the underlying-set functor, and also admits finite colimits: the pushout in $\mathcal{P}$ of $f\colon A \to B$ and $g\colon A \to C$ is given by $B \otimes_A C$, while their pushout in $\mathcal{Q}$ is the quotient of this by its radical R, consisting of the nilpotent elements; the initial object either in $\mathcal{P}$ or in $\mathcal{Q}$ is of course given by k itself.

For an object P of $\mathcal{P}$, the atoms among its idempotents induce an (essentially unique) expression

$$P = P_1 \times P_2 \times \cdots \times P_n \tag{9.1}$$

of P as a product of indecomposables; when P has zero radical and thus is semi-simple, so too is each P_j. If $Q \in \mathcal{P}$ has a similar expression

$$Q = Q_1 \times Q_2 \times \cdots \times Q_m, \tag{9.2}$$

a map $f\colon P \to Q$ in $\mathcal{P}$ is of course given by components $\pi_i f\colon P \to Q_i$ for $1 \leqslant i \leqslant m$, where the $\pi_i\colon Q \to Q_i$ are the projections. The atomic idempotent e_j in P corresponding to P_j is taken by $\pi_i f$ to some idempotent in Q_i, which must be 0 or 1; and because $e_j e_k = 0$ for $j \neq k$ while $\sum e_j = 1$, there is for each i exactly one j – we may call it ϕi – for which $(\pi_i f)(e_j)$ is non-zero. It follows that $\pi_i f$ is a composite

$$P_1 \times \cdots \times P_n \xrightarrow[\pi_{\phi i}]{} P_{\phi i} \xrightarrow[f_i]{} Q_i\ . \tag{9.3}$$

In summary, then, a map $f\colon P \to Q$ in $\mathcal{P}$ is given by a function $\phi\colon m \to n$ together with maps $f_i\colon P_{\phi i} \to Q_i$ for $1 \leqslant i \leqslant m$.

The indecomposable algebras are those whose only idempotents are 0 and 1; they are the algebras that are local rings, and they form a category $\mathcal{L}$. Those which are also semi-simple are just the fields K that are finite extensions of k;

they form a category $\mathcal{K}$. We can express the description above of the objects and the maps by writing

$$\mathcal{P}^{\mathrm{op}} \simeq \mathrm{Fam}\,\mathcal{L}^{\mathrm{op}}, \qquad \mathcal{Q}^{\mathrm{op}} \simeq \mathrm{Fam}\,\mathcal{K}^{\mathrm{op}}, \tag{9.4}$$

where Fam $\mathcal{A}$ denotes the category of *finite families of objects of the category* $\mathcal{A}$ – a concept that we now recall.

9.2. Given any category $\mathcal{A}$, we have a category Fam $\mathcal{A}$ of *finite families* of objects of $\mathcal{A}$; we shall denote it here by $\mathcal{C}$, so that in the examples of (9.4) our category $\mathcal{C}$ is either $\mathcal{P}^{\mathrm{op}}$ or $\mathcal{Q}^{\mathrm{op}}$. (Of course one could deal equally with *infinite* families; it is just that we use below only finite ones.)

An object of $\mathcal{C} = \mathrm{Fam}\,\mathcal{A}$ is a pair (X, A) where X is a finite set (which we may see as a discrete category) and $A\colon\ X \to \mathcal{A}$ is a functor; that is to say, $(A_x)_{x\in X}$ is an X-indexed family of objects of $\mathcal{A}$. A map $(X, A) \to (Y, B)$ in $\mathcal{C}$ is a pair (ϕ, f), where $\phi\colon\ X \to Y$ is a function and $f\colon\ A \to B\phi\colon\ X \to \mathcal{A}$ is a (natural) transformation; that is, f consists of maps $f_x\colon\ A_x \to B_{\phi x}$ in $\mathcal{A}$ for $x \in X$. These maps in $\mathcal{C}$ are composed in the obvious way.

It is often of course convenient to denote an object (X, A) of $\mathcal{C}$ by a single letter such as C, and equally to denote a map $(\phi, f)\colon\ (X, A) \to (Y, B)$ in $\mathcal{C}$ by a single letter such as $h\colon\ C \to D$. Where confusion is unlikely, it may be intuitively helpful to denote (X, A), thought of as the family $(A_x)_{x\in X}$, by the single letter A itself. It is otherwise for maps: the risk of misunderstanding in writing f for (ϕ, f) so much outweighs any slight gain in brevity of expression that we rarely do so – diagram (9.13) below being an exception.

The category $\mathcal{C}$ admits finite coproducts: the initial object is the unique (X, A) with X empty, while $(X, A) + (Y, B)$ is given by $(A, B)\colon\ X + Y \to \mathcal{A}$. The *indecomposable objects* of $\mathcal{C}$ – those that cannot be written non-trivially as a coproduct – are those of the form $(1, K)$ where $K \in \mathcal{A}$; they form a full subcategory of $\mathcal{C}$ equivalent to $\mathcal{A}$ itself, and often identified with it. In some contexts, it is natural to call the indecomposable objects *connected*. Each $(Y, B) \in \mathcal{C}$ admits an essentially unique decomposition

$$(Y, B) = \sum_{y\in Y} (1, B_y) \tag{9.5}$$

as a coproduct of indecomposables. When we denote (Y, B) by the single letter B, as suggested above, and identify $\mathcal{A}$ with its image in $\mathcal{C}$, we may write (9.5) as

$$B = \sum_{y\in Y} B_y; \tag{9.6}$$

but of course (9.6) may well represent a coproduct in $\mathcal{C}$ wherein the B_y are not indecomposables but arbitrary objects.

The category $\mathcal{C} = \mathrm{Fam}\mathcal{A}$ is easily seen to be what is called an *extensive* category in the language of Carboni, Lack, and Walters [5]; by this is meant that, for *any* finite coproduct (9.6), the functor sending a family $(f_y\colon A_y \to B_y)_{y\in Y}$ of maps to their coproduct $\sum f_y\colon \sum A_y \to \sum B_y$ is an equivalence of categories

$$\prod_{y\in Y} \mathcal{C}/B_y \;\simeq\; \mathcal{C}/B. \tag{9.7}$$

As these authors show, it suffices to require this when Y has just two elements.

In particular, returning to the case where (9.6) is the canonical decomposition (9.5) of (Y, B), an object $(\phi, f)\colon (X, A) \to (Y, B)$ of the right side of (9.7) is (essentially) uniquely expressible in the form

$$(\phi, f) = \sum_{y\in Y} g_y\colon \sum_{y\in Y} C_y \to \sum_{y\in Y} B_y; \tag{9.8}$$

we have only to set

$$C_y = \sum_{\phi x = y} A_x, \tag{9.9}$$

and to take $g_y\colon C_y \to B_y$ to be the map given by

$$(g_y)_x = f_x. \tag{9.10}$$

9.3. We intend to develop the theory as far as we can for a general category of the form $\mathcal{C}$ = Fam $\mathcal{A}$, before returning in Section 9.12 to our examples (9.4). To make progress, we need information about *pullbacks* in a category of the form Fam $\mathcal{A}$. For a general $\mathcal{A}$, this need not admit pullbacks at all; but our examples of $\mathcal{C}$ in (9.4) certainly have pullbacks.

We first observe that we can describe all pullbacks in $\mathcal{C}$ very simply in terms of pullbacks of maps between indecomposables. Suppose we have in $\mathcal{C}$ maps $(\phi, f)\colon (X, A) \to (Y, B)$ and $(\psi, g)\colon (Z, C) \to (Y, B)$. First form, in **Set**, the pullback

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "\phi"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd} \quad , \tag{9.11}$$

and write a typical element of W as a triple $w = (x, y, z)$ where $\phi x = y = \psi z$. Now, for each such w, consider in $\mathcal{C}$ the pullback

$$\begin{tikzcd} D_w \arrow[r, "k_w"] \arrow[d, "h_w"'] & A_x \arrow[d, "f_x"] \\ C_z \arrow[r, "g_y"'] & B_y \end{tikzcd} , \tag{9.12}$$

where A_x, B_y, and C_z are of course indecomposables; in general D_w will not be indecomposable. However we can form in $\mathcal{C}$ the coproduct $D = \sum_{w \in W} D_w$; and we have in $\mathcal{C}$ the diagram

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd} , \tag{9.13}$$

where A, B, and C stand for (X, A), (Y, B), and (Z, C), while f and g stand for (ϕ, f) and (ψ, g), and h and k denote the evident maps in $\mathcal{C}$ given by the h_w and the k_w. We leave to the reader the easy verification – it suffices to use an indecomposable as a test-object – of the fact that:

The diagram (9.13) is a pullback in $\mathcal{C}$. (9.14)

It may have mnemonic value to formulate this, albeit somewhat imprecisely, as

$$\sum A_x \times_{\sum B_y} \sum C_z \;=\; \sum A_x \times_{B_y} C_z, \tag{9.15}$$

it being understood that the A_x, B_y, C_z here are indecomposables.

9.4. Given (9.14), the reader will find it easy to verify that, if

$$\begin{tikzcd} D_x \arrow[r, "k_x"] \arrow[d, "h_x"'] & A_x \arrow[d, "f_x"] \\ C_x \arrow[r, "g_x"'] & B_x \end{tikzcd} \tag{9.16}$$

is, for each x in the finite set X, a pullback diagram in $\mathcal{C}$ (whose vertices are not at all required to be indecomposable), then so too is the diagram

$$\begin{tikzcd} \sum D_x \arrow[r, "\sum k_x"] \arrow[d, "\sum h_x"'] & \sum A_x \arrow[d, "\sum f_x"] \\ \sum C_x \arrow[r, "\sum g_x"'] & \sum B_x \end{tikzcd} \quad . \tag{9.17}$$

9.5. The reader will also easily verify that, for every $g\colon C \to B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} C + C + \cdots + C \arrow[r, "g+g+\cdots+g"] \arrow[d, "\delta"'] & B + B + \cdots + B \arrow[d, "\delta"] \\ C \arrow[r, "g"'] & B \end{tikzcd} \quad , \tag{9.18}$$

wherein each δ is a co-diagonal, is a pullback.

Suppose now that we have maps $f_x\colon A_x \to B$ for $x \in X$, and that for each $x \in X$ we have a pullback

$$\begin{tikzcd} D_x \arrow[r, "k_x"] \arrow[d, "h_x"'] & A_x \arrow[d, "f_x"] \\ C \arrow[r, "g"'] & B \end{tikzcd} \quad . \tag{9.19}$$

By Section 9.4, these pullbacks give rise to a pullback of the form (9.17), wherein however each $g_x\colon C_x \to B_x$ is a copy of $g\colon C \to B$; pasting this on top of (9.18) produces a pullback

$$\begin{tikzcd} \sum D_x \arrow[r, "\sum k_x"] \arrow[d, "h = (h_x)"'] & \sum A_x \arrow[d, "f = (f_x)"] \\ C \arrow[r, "g"'] & B \end{tikzcd} \quad . \tag{9.20}$$

If now in (9.20) we take C to be $\sum C_z$ and g to be (g_z) where $g_z\colon\ C_z \to B$, and then use again the result of (9.20), but now with g replacing f, we conclude that

$$\sum A_x \times_B \sum C_z \;=\; \sum_{x,z} A_x \times_B C_z. \tag{9.21}$$

Finally, if we replace B here by a coproduct $\sum B_y$, and take f and g to be given by maps $f_x\colon\ A_x \to B_{\phi x}$ and $g_z\colon\ C_z \to B_{\psi z}$ for suitable ϕ and ψ, we re-find (9.15) – but now with no requirement that the A_x, B_y, C_z be indecomposable.

9.6. We have seen that the indecomposable objects of $\mathcal{C} = \mathrm{Fam}\,\mathcal{A}$ – those of the form $(1, K)$ where $K \in \mathcal{A}$ – form a full subcategory equivalent to $\mathcal{A}$. There is another full subcategory $\mathcal{X}$ of $\mathcal{C}$ formed by the *discrete* objects (which might also be called the *totally disconnected* objects): these are the objects $(X, \Delta 1)$, where $\Delta 1\colon\ X \to \mathcal{A}$ is the constant functor at the terminal object 1 of $\mathcal{A}$ (which certainly exists in our examples (9.4), the initial object k of $\mathcal{P}$ lying in $\mathcal{K}$ and hence in $\mathcal{L}$). In fact $\mathcal{X}$ is nothing but the category of finite sets, the functor $X \mapsto (X, \Delta 1)$ providing the identification.

The inclusion $H\colon\ \mathcal{X} \to \mathcal{C}$ clearly has a left adjoint I, with $I(X, A) = (X, \Delta 1)$, and with $\eta_{(X,A)}\colon\ (X, A) \to I(X, A) = (X, \Delta 1)$ given by the identity $X \to X$ and the unique maps !: $A_x \to 1$; so that

$$\eta_{(X,A)} = (1_X, !)\colon\ (X, A) \to (X, \Delta 1). \tag{9.22}$$

We write $(\mathcal{E}, \mathcal{M})$ for the reflective prefactorization system $\Phi\mathcal{X}$ on $\mathcal{C}$ given by this reflexion, noting that (3.2) gives:

$$\begin{aligned} &(\phi, f)\colon\ (X, A) \to (Y, B) \quad \text{lies in } \mathcal{E} \text{ if and only if} \\ &\quad \phi\colon\ X \to Y \quad \text{is bijective.} \end{aligned} \tag{9.23}$$

For the coproduct $\sum f_x\colon\ \sum A_x \to \sum B_x$ of a finite family $(f_x\colon\ A_x \to B_x)_{x \in X}$ of maps in $\mathcal{C}$, we have

$$\sum f_x \text{ lies in } \mathcal{E} \text{ if and only if each } f_x \text{ lies in } \mathcal{E}; \tag{9.24}$$

the "if" part is true for any prefactorization system, by Proposition 2.2(d), but here (9.23) gives "only if" as well.

It follows easily from Section 9.3 that, for a general $\phi = (\phi, \Delta 1)\colon\ (X, \Delta 1) \to (Y, \Delta 1)$ in $\mathcal{X}$, we have in $\mathcal{C}$ the pullback

$$\begin{tikzcd}
(X, D) \arrow[r, "{(1_X, !)}"] \arrow[d, "{(\phi, 1)}"'] & (X, \Delta 1) \arrow[d, "{(\phi, \Delta 1)}"] \\
(Y, B) \arrow[r, "{(1_Y, !)}"'] & (Y, \Delta 1)
\end{tikzcd} \tag{9.25}$$

,

wherein $D_x = B_{\phi x}$. Since the top edge of (9.25) is in $\mathcal{E}$ by (9.23), we conclude that the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ is admissible in the sense of Section 3.6; accordingly, by Section 3.5, the prefactorization system $(\mathcal{E}, \mathcal{M})$ is a factorization system.

In contrast to the examples in Sections 7 and 8 above, this reflexion does *not* have stable units in general. For consider the pullback (3.11) in our $\mathcal{C}$, where $Z = 1$ and B, C are indecomposable; to say that I preserves this pullback is to say that $B \times C$ is indecomposable; but this is generally false in our examples (9.4).

9.7. Using the calculation in Section 9.3 of pullbacks in $\mathcal{C}$ (in a case where the basic pullbacks (9.12) are trivial), we find the diagram (3.5) in the present case to be:

$$\begin{tikzcd}
(X,A) \arrow[dr, "{(1,f)}"] \arrow[drr, "{(1,!)}"] \arrow[ddr, "{(\phi,f)}"'] & & \\
& (X,B\phi) \arrow[r, "{(1,!)}"'] \arrow[d, "{(\phi,1)}"] & (X,\Delta 1) \arrow[d, "{(\phi,\Delta 1)}"] \\
& (Y,B) \arrow[r, "{(1,!)}"'] & (Y,\Delta 1)
\end{tikzcd} \quad . \tag{9.26}$$

It follows from Sections 3.6 and 3.5 that

$$(\phi, f) = (\phi, 1)(1, f) \tag{9.27}$$

is the $(\mathcal{E}, \mathcal{M})$-factorization of (ϕ, f). Since (ϕ, f) lies in $\mathcal{M}$ precisely when its $\mathcal{E}$-part $(1, f)$ in (9.27) is invertible, we conclude that

$$\begin{aligned} &(\phi, f)\colon\ (X, A) \to (Y, B) \quad \text{lies in } \mathcal{M} \text{ if and} \\ &\qquad \text{only if each} \quad f_x\colon\ A_x \to B_{\phi x} \quad \text{is invertible.} \end{aligned} \tag{9.28}$$

If we again consider a family $(f_x\colon\ A_x \to B_x)_{x \in X}$ of maps in $\mathcal{C}$, and its coproduct $\sum f_x\colon\ \sum A_x \to \sum B_x$, we conclude from (9.28) that

$$\sum f_x \text{ lies in } \mathcal{M} \text{ if and only if each } f_x \text{ lies in } \mathcal{M}. \tag{9.29}$$

Equally, for a family of maps $f_x\colon\ A_x \to B$, (9.28) gives:

$$\text{a map}\quad f = (f_x)\colon \textstyle\sum A_x \to B \quad \text{lies in } \mathcal{M} \text{ if and only if each } f_x \text{ does so.} \tag{9.30}$$

9.8. Let us now determine the class $\mathcal{E}'$. This class $\mathcal{E}'$ is a subclass of $\mathcal{E}$, and $(\phi, f)\colon\ (X, A) \to (Y, B)$ lies in $\mathcal{E}$, as we have seen in (9.23), if and only if ϕ is bijective; accordingly we may as well simplify by considering a map $(1, f)\colon\ (X, A) \to (X, B)$ given by components $f_x\colon\ A_x \to B_x$, these A_x and B_x being of course objects of $\mathcal{A}$, and hence indecomposable. This is a circumstance in which it is particularly useful to write $(1, f)\colon\ (X, A) \to (X, B)$ as $\sum f_x\colon\ \sum A_x \to \sum B_x$, and to call it $f\colon\ A \to B$.

It is immediate from Section 9.3 that $f_x\colon\ A_x \to B_x$ is the pullback along the injection $B_x \to \sum B_x$ of $f = \sum f_x\colon\ \sum A_x \to \sum B_x$; since $\mathcal{E}'$ is pullback-stable, it follows that $f_x \in \mathcal{E}'$ for each x when $f \in \mathcal{E}'$. The converse, however, is also true: any $g\colon\ C \to B$ is by (9.8) of the form $\sum g_x\colon\ \sum C_x \to \sum B_x$, so that by (9.17) the pullback of f by g is $\sum g_x^*(f_x)$, which lies in $\mathcal{E}$ by (9.24) when each f_x lies in $\mathcal{E}'$.

It remains therefore to determine which of the maps $f\colon\ A \to B$, where both A and B are indecomposable, lie in $\mathcal{E}'$. For this it is clearly necessary, by (9.23), that *in each pullback*

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

where C too is indecomposable, we have D indecomposable. In fact this condition is also sufficient, since a general map g into the indecomposable B has the form $(g_x)\colon\ \sum C_x \to B$ with each C_x indecomposable, and by (9.20) the pullback $g^*(f)$ is $\sum g_x^*(f)$ – which by (9.24) lies in $\mathcal{E}$ when each of its summands does so.

It follows of course from the second paragraph of this section that, for *any* family of maps $f_x\colon\ A_x \to B_x$, we have

$$\textstyle\sum f_x \text{ lies in } \mathcal{E}' \text{ if and only if each } f_x \text{ lies in } \mathcal{E}'. \tag{9.31}$$

9.9. We shall need the observation that, if the maps $p_x\colon\ E_x \to B_x$ are effective descent maps in $\mathcal{C}$ for each $x \in X$, so too is $p = \sum p_x\colon\ \sum E_x \to \sum B_x$. The point is that the diagram

$$\begin{tikzcd}
\mathcal{C}/\sum B_x \arrow[r, "p^*"] & \mathcal{C}/\sum E_x \\
\prod(\mathcal{C}/B_x) \arrow[u, "+"] \arrow[r, "\prod p_x^*"'] & \prod(\mathcal{C}/E_x) \arrow[u, "+"']
\end{tikzcd}$$

commutes to within isomorphism by (9.17), while the vertical arrows are equivalences by (9.7); so that p^* like $\prod p_x^*$ is monadic.

9.10. We turn to consideration of the $f\colon\ A \to B$ in $\mathcal{M}^*$, reducing the question to the case of indecomposable A and B:

PROPOSITION. *The map* $(\phi, f)\colon\ (X, A) \to (Y, B)$ *of* $\mathcal{C}$ *lies in* $\mathcal{M}^*$ *if and only if each* $f_x\colon\ A_x \to B_{\phi x}$ *does so.*

Proof. First, we can use (9.8) to write f as $\sum g_y\colon\ \sum C_y \to \sum B_y$, with the B_y of course indecomposable. If f here lies in $\mathcal{M}^*$, so too does each g_y; for $\mathcal{M}^*$ is pullback-stable by Section 6.1, and g_y is by Section 9.3 the pullback of f along the coproduct-injection $(1, B_y) \to (Y, B)$ of (9.5). Conversely, f lies in $\mathcal{M}^*$ when each g_y does so; for if $p_y\colon\ E_y \to B_y$ are effective descent maps with each $p_y^*(g_y)$ in $\mathcal{M}$, then $p = \sum p_y$ is an effective descent map by Section 9.9, while (9.17) tells us that $p^*(f) = \sum p_y^*(g_y)$, and the latter lies in $\mathcal{M}$ by (9.29).

We are thus reduced to the case of a map $f = (f_x)\colon\ \sum A_x \to B$ in $\mathcal{C}$, where B and the A_x are indecomposable. If f lies in $\mathcal{M}^*$, so too does each f_x; for by (9.20) the pullback $p^*(f)$ along an effective descent map p has components $p^*(f_x)$, and these lie in $\mathcal{M}$ by (9.30) when $p^*(f)$ does so. Conversely, f lies in $\mathcal{M}^*$ when each f_x does so. For let $p_x\colon\ E_x \to B$ be effective descent maps with $p_x^*(f_x) \in \mathcal{M}$, and let $p\colon\ E \to B$ be the fibred product of the p_x, which is an effective descent map by (b) and (c) of Section 4.7. Then, since $\mathcal{M}$ is pullback-stable by Proposition 2.2, each $p^*(f_x) = (p^*(f))_x$ lies in $\mathcal{M}$, whence $p^*(f) \in \mathcal{M}$ by (9.30), so that $f \in \mathcal{M}^*$.

9.11. We return now to the examples $\mathcal{C} = \mathcal{P}^{\mathrm{op}}$ and $\mathcal{C} = \mathcal{Q}^{\mathrm{op}}$ of Section 9.1. The basic results on separable and purely inseparable field extensions can be found in van der Waerden [24, Ch. IV, Section 44], or in the later Jacobson [11, Ch. I, Sections 8 and 9]; these results are of course non-trivial only when the characteristic π of our base-field k is non-zero, since otherwise *every* finite extension is separable. We recall the following well-known result:

LEMMA. *If* A *is a separable extension field of the field* B*, while* C *is a semisimple* B*-algebra, then the* B*-algebra* $A \otimes_B C$ *is also semisimple.*

Proof. It suffices to consider the case where C is indecomposable, and is thus itself an extension field of B. The separable extension A of B is a simple extension $B(a)$, and therefore has the form $B[x]/\phi(x)B[x]$ where $\phi[x]$ is an irreducible polynomial with distinct zeros α_i. We are to show that $A \otimes_B C$ has no nilpotents; and since $(A \otimes_B -)$ preserves monomorphisms – see, for this, the proof of Proposition 9.12 below – it suffices to show that $A \otimes_B E$ has no nilpotents, where E is some extension field of C. We so choose E that $\phi(x)$ decomposes in $E[x]$ into linear factors; then

$$A \otimes_B E \cong E[x]/\phi(x)E[x] \cong \prod(E[x]/(x-\alpha_i)E[x]),$$

and this has no nilpotents since $E[x]/(x-\alpha_i)E[x] \cong E$.

9.12. PROPOSITION. *If the map $f\colon\ B \to A$ of $\mathcal{Q}$ is the inclusion of a field B into a separable extension field A, then $f\colon\ A \to B$ is an effective descent map in each of $\mathcal{P}^{\mathrm{op}}$ and $\mathcal{Q}^{\mathrm{op}}$.*

Proof. In each case we are to prove $f^*\colon\ \mathcal{C}/B \to \mathcal{C}/A$ monadic. When $\mathcal{C} = \mathcal{P}^{\mathrm{op}}$, the functor f^* is

$$(A \otimes_B -)^{\mathrm{op}}\colon\ (B - \mathrm{Alg})^{\mathrm{op}} \to (A - \mathrm{Alg})^{\mathrm{op}}; \tag{9.32}$$

so that it suffices by Beck's monadicity results (see in particular Mac Lane [20, Ch. VI, Section 7, Ex. 3]) to show that $(A \otimes_B -)$ is left exact and reflects isomorphisms. In fact it suffices to show that the composite $U(A\otimes_B -)$ has these properties, where U is the forgetful functor A-Alg $\to$ B-Mod; for U preserves and reflects finite limits. This is indeed the case, since A as a B-module is a direct sum $n \cdot B$ of copies of B, so that $A \otimes_B C$ as a B-module is just the direct sum $n \cdot C$ of n copies of C.

So far, we have not used the separability of A; but we need it, via its consequence Lemma 9.11, in the case $\mathcal{C} = \mathcal{Q}^{\mathrm{op}}$. There, by Lemma 9.11, the functor f^* is merely the restriction of (9.32) to the semisimple algebras. This surely reflects isomorphisms; moreover the restriction of $(A \otimes_B -)$ is left exact since the semisimple B-algebras are reflective in B-Alg and hence closed under limits, and similarly for A-algebras.

9.13. PROPOSITION. *If the map $f\colon\ B \to A$ of $\mathcal{Q}$ is the inclusion of a field B into a separable extension field A, then the map $f\colon\ A \to B$ of $\mathcal{Q}^{\mathrm{op}}$ lies in $\mathcal{M}^*$.*

Proof. Let $A = B(a) \cong B[x]/\phi(x)B[x]$ as in the proof of Lemma 9.11, and let $p\colon\ B \to E$ be the inclusion of B into the splitting field of $\phi(x)$; then, since p is a separable extension, $p\colon\ E \to B$ is by Proposition 9.12 an effective descent map in $\mathcal{Q}^{\mathrm{op}}$. The pullback $p^*(f)$ in $\mathcal{Q}^{\mathrm{op}}$ is the corresponding pushout in $\mathcal{Q}$, which by Lemma 9.11 is $E \to A \otimes_B E$; and (by the argument in the proof of Lemma 9.11) this is a diagonal map $E \to E \times E \times \cdots \times E$ in $\mathcal{Q}$, so that in $\mathcal{Q}^{\mathrm{op}}$ it lies in $\mathcal{M}$ by (9.28).

9.14. PROPOSITION. *The map $f\colon B \to A$ of $\mathcal{Q}$ given by a field extension, seen as a map $f\colon A \to B$ in $\mathcal{Q}^{\mathrm{op}}$, lies in $\mathcal{E}'$ if and only if the extension is purely inseparable.*

Proof. For the "if" part it suffices – since $\mathcal{E}'$ is closed under composition by Section 2.11 – to suppose that A is a simple extension $B[x]/\psi(x)B[x]$, where the irreducible $\psi(x)$ has the form $x^{\pi^e} - b$ for some $b \in B$. By Section 9.8, to show that $f \in \mathcal{E}'$ is to show that, for every field extension $g\colon B \to C$, the pushout $(A \otimes_B C)/R$ in $\mathcal{Q}$ (where R is the radical) is indecomposable; for which it suffices that $A \otimes_B C$ be indecomposable – or equally that it contain no non-trivial idempotent. Without loss of generality we may replace C by a further extension D over which $\psi(x)$ decomposes into linear factors – so that $\psi(x)$ has the form $(x-d)^{\pi^e}$; now

$$\begin{aligned} A \otimes_B D &\cong (B[x]/\psi(x)B[x]) \otimes_B D \cong D[x]/\psi(x)D[x] \\ &= D[x]/(x-d)^{\pi^e} D[x] \cong D[x]/x^{\pi^e} D[x], \end{aligned}$$

and this clearly has no non-trivial idempotent.

For the converse, let C be the separable closure of B in A, so that f is the composite of the separable extension $g\colon B \to C$ and the purely inseparable extension $h\colon C \to A$. Let $C = B(a) \cong B[x]/\phi(x)B[x]$ where the irreducible $\phi(x)$ has the distinct zeros α_i for $1 \leqslant i \leqslant n$; and write $p\colon B \to E$ for the inclusion of B into the splitting field E of $\phi(x)$. Because $f \in \mathcal{E}'$, it follows from Section 9.8 that the pushout in $\mathcal{Q}$ of f and p – which by Lemma 9.11 is $E \otimes_B A$ since p is a separable extension – has no non-trivial idempotents. Thus, since $E \otimes_B h\colon E \otimes_B C \to E \otimes_B A$ is monomorphic, $E \otimes_B C$ has no non-trivial idempotents. As in the proof of Lemma 9.11, however, $E \otimes_B C$ is the product of n copies of E; it follows that $n = 1$, so that in fact $C = B$ and the extension $f\colon B \to A$ is purely inseparable.

9.15. THEOREM. *On $\mathcal{Q}^{\mathrm{op}}$, the classes $\mathcal{E}'$ and $\mathcal{M}^*$ constitute a factorization system $(\mathcal{E}', \mathcal{M}^*)$. The map $(\phi, f)\colon (X, A) \to (Y, B)$ of $\mathcal{Q}^{\mathrm{op}}$ lies in $\mathcal{E}'$ precisely when ϕ is a bijection and each $f_x\colon B_{\phi x} \to A_x$ is a purely inseparable extension in $\mathcal{Q}$; and it lies in $\mathcal{M}^*$ precisely when each f_x is a separable extension. The $(\mathcal{E}', \mathcal{M}^*)$-factorization in $\mathcal{Q}^{\mathrm{op}}$ of a general (ϕ, f) is*

$$(X, A) \xrightarrow[(1,h)]{} (X, C) \xrightarrow[(\phi,g)]{} (Y, B), \tag{9.33}$$

where

$$B_{\phi x} \xrightarrow[g_x]{} C_x \xrightarrow[h_x]{} A_x \tag{9.34}$$

is the decomposition in $\mathcal{Q}$ of the field extension $f_x\colon B_{\phi x} \to A_x$ into a separable extension g_x and a purely inseparable extension h_x, formed by taking for C_x the separable closure of $B_{\phi x}$ in A_x.

Proof. That $\mathcal{E}'$ consists precisely of the maps described above follows from Section 9.8 and Proposition 9.14. That the maps (ϕ, f) with each f_x a separable extension do lie in $\mathcal{M}^*$ follows from Propositions 9.10 and 9.13. Thus (9.33) provides an $(\mathcal{E}', \mathcal{M}^*)$ -factorization for a general map (ϕ, f). As we observed in the second paragraph of Section 6.9, this is all that we need – in the light of Proposition 6.7 – to conclude that $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system. It remains to show that the *only* maps (ϕ, f) in $\mathcal{M}^*$ are those with each f_x: $B_{\phi x} \to A_x$ separable; but this follows from (9.33) and (9.34), since (ϕ, f) lies in $\mathcal{M}^*$ precisely when its $\mathcal{E}'$-part $(1, h)$ in (9.33) is invertible; that is, when all the h_x of (9.34) are invertible, so that all the f_x are separable.

9.16. REMARK. Similar but longer calculations, which we leave to the reader, give a corresponding result for $\mathcal{P}^{\mathrm{op}}$. Once again $(\mathcal{E}', \mathcal{M}^*)$ is a factorization system. Write RA for the radical of the algebra A, and SA for its semisimple quotient A/RA. Then the map (ϕ, f): $(X, A) \to (Y, B)$ of $\mathcal{P}^{\mathrm{op}}$ lies in $\mathcal{E}'$ precisely when ϕ is a bijection and each Sf_x: $SB_{\phi x} \to SA_x$ is a purely inseparable extension in $\mathcal{Q}$; while (ϕ, f) lies in $\mathcal{M}^*$ precisely when each Rf_x: $RB_{\phi x} \to RA_x$ is an isomorphism and each Sf_x: $SB_{\phi x} \to SA_x$ is a separable extension in $\mathcal{Q}$.

9.17. REMARK. There is a factorization system $(\mathcal{F}, \mathcal{N})$ on $\mathcal{P}^{\mathrm{op}}$ for which $\mathcal{N}$ consists of the surjections in $\mathcal{P}$ and $\mathcal{F}$ of the monomorphisms in $\mathcal{P}$; so that $\mathcal{F}$ consists of the epimorphisms in $\mathcal{P}^{\mathrm{op}}$, and $\mathcal{N}$ of the regular monomorphisms there. This gives us, as in Section 3.9, a factorization system $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$, where $\overline{\mathcal{E}} = \mathcal{E} \cap \mathcal{F}$. For our example of compact Hausdorff spaces, we found in Section 7.6 that $(\overline{\mathcal{E}}', \overline{\mathcal{M}}^*)$ was a factorization system coinciding with $(\mathcal{E}', \mathcal{M}^*)$; while for our example of hereditary torsion theories with the projectives torsion-free, we found in Theorem 8.13 that $(\mathcal{E}', \mathcal{M}^*)$ was $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ itself – which then, as in Remark 6.12, coincides with $(\overline{\mathcal{E}}', \overline{\mathcal{M}}^*)$. In the case of $\mathcal{P}^{\mathrm{op}}$, the situation is quite different: although $(\overline{\mathcal{E}}', \overline{\mathcal{M}}^*)$ too turns out to be a factorization system, it fails to coincide with $(\mathcal{E}', \mathcal{M}^*)$. In fact, as the reader may verify, the map (ϕ, f): $(X, A) \to (Y, B)$ of $\mathcal{P}^{\mathrm{op}}$ lies in $\overline{\mathcal{M}}^*$ precisely when each Sf_x: $SB_{\phi x} \to SA_x$ is a separable extension in $\mathcal{Q}$ and each Rf_x: $RB_{\phi x} \to RA_x$ is a *surjection* in $\mathcal{P}$.

When we return to $\mathcal{Q}$, however, the radicals are all zero, and once again $(\overline{\mathcal{E}}', \overline{\mathcal{M}}^*)$ coincides with $(\mathcal{E}', \mathcal{M}^*)$.

10. The Counter-Examples

10.1. We turn now to the counter-examples promised in Section 6.1. For the first, we are to produce a factorization system $(\mathcal{E}, \mathcal{M})$, of the form $\Phi\mathcal{X}$ for an admissible reflexion of $\mathcal{C}$ onto a full subcategory $\mathcal{X}$, for which $\mathcal{M}^*/B$ is not reflective in $\mathcal{C}/B$.

We take for $\mathcal{C}$ the category of sheaves on a connected and locally-connected topological space P, thinking of such a sheaf A as a space with a local homeomorphism $A \to P$; note that A like P is locally connected. Since pullbacks of local homeomorphisms are local homeomorphisms in the category **Top** of topological spaces, pullbacks in $\mathcal{C}$ are formed as in **Top**.

It is convenient to identify a set S with the corresponding discrete space, and then to identify it further with the *constant sheaf* given by the projection $S \times P \to P$; we have here an isomorphism of the category of sets with the full subcategory $\mathcal{X}$ of $\mathcal{C}$ given by the constant sheaves. If πA denotes the set (or discrete space) of components of $A \in \mathcal{C}$, the canonical map $A \to \pi A$ in **Top** is continuous, and the corresponding map $\eta_A\colon\ A \to \pi A \times P$ in $\mathcal{C}$ is the unit of a reflexion I of $\mathcal{C}$ onto $\mathcal{X}$.

Thus a map $f\colon\ A \to B$ in $\mathcal{C}$ lies in $\mathcal{E}$ precisely when it induces a bijection between the components of A and those of B; it follows easily from this that the reflexion of $\mathcal{C}$ onto $\mathcal{X}$ is admissible in the sense of Section 3.6, and *a fortiori* is simple. Accordingly $f\colon\ A \to B$ lies in $\mathcal{M}$ precisely when (3.9) is a pullback – which is to say that f is a sum of projections $\sum S_i \times B_i \to \sum B_i$, where the B_i are the components of B and the S_i are discrete; as in Section 5.1, we call such an f a *trivial covering* of B. Of course a trivial covering of a connected B, such as P itself, is a constant sheaf $S \times B \to B$.

The effective descent maps in the topos $\mathcal{C}$, which is of course an exact category, are the epimorphisms by (a) of Section 4.7. It follows at once that the maps in $\mathcal{M}^*/B$ – which are those we called the *coverings* of B in Section 5.1 – are just the coverings in the ordinary sense of this word: a map $f\colon\ A \to B$ is a covering if each $b \in B$ has an open neighbourhood U over which the restriction $f^*(U) \to U$ of f has the form of a projection $S \times U \to U$ for some discrete space S.

A connected object B of $\mathcal{C}$ is said to be *simply connected* if $\mathcal{M}^*/B = \mathcal{M}/B$; that is, if every covering of B is trivial. Moreover an object B is said to be *locally simply connected* if each $b \in B$ has a simply-connected open neighbourhood.

Barr and Diaconescu show, in [1, Thm. 6], that $\mathcal{M}^*/P$ is reflective here in $\mathcal{C}/P$ if and only if there exists an epimorphism $p\colon E \to P$ such that $\mathcal{M}^*/P = \mathrm{Spl}(E, p)$, in the sense of Section 5.1. In their discussion following this theorem, they further point out that this reflectivity obtains whenever P is locally simply connected; and that, when it does obtain, P has a simply-connected universal covering (by which is meant a connected and weakly-initial object in $\mathcal{M}^*/P$).

If we take for P a locally-connected space which lacks a universal covering, such as the union in the plane, for $n \in \mathbf{N}$, of the circles of centre $(0, 1/n)$ and radius $1/n$, we have our desired counter-example where $\mathcal{M}^*/P$ is not reflective in $\mathcal{C}/P$.

10.2. Let us again take $\mathcal{C}$ as in Section 10.1, but now with P the simply-connected space given by the unit disk $\{(x, y)|x^2 + y^2 \leqslant 1\}$ in the plane. Here every $B \in \mathcal{C}$

is locally simply connected, so that each $\mathcal{M}^*/B$ is reflective in $\mathcal{C}/B$ by the result above of Barr and Diaconescu. In this $\mathcal{C}$, moreover, $\mathcal{M}^*$ is closed under composition. For suppose that $f\colon\ A \to B$ and $g\colon\ B \to C$ lie in $\mathcal{M}^*$, and let $c \in C$. If U is a simply-connected open neighbourhood of c in C, the covering $g^*(U) \to U$ has the form of a projection $S \times U \to U$ for a discrete S; while for $s \in S$ the covering $f^*(\{s\} \times U) \to \{s\} \times U$ has the form of a projection $T_s \times \{s\} \times U \to \{s\} \times U$ for a discrete T_s. So the map $(gf)^*(U) \to U$ is the projection $R \times U \to U$ where $R = \sum_{s \in S} T_s$, and gf is indeed a covering.

By Section 2.12, therefore, $(\mathcal{M}^{*\uparrow}, \mathcal{M}^*)$ is a factorization system. We shall now observe that $\mathcal{M}^{*\uparrow} \neq \mathcal{E}'$, so that we have the third of the counter-examples promised in Section 6.1.

Let A be the subset $\{(x, y) \mid 0 < y \leqslant 1\}$ of the plane, and let $f\colon\ A \to P$ be given by $f(x, y) = (y \cos x, y \sin x)$; clearly f is a local homeomorphism and hence a map in $\mathcal{C}$. Suppose that the reflexion into $\mathcal{M}^*/P$ of the object f of $\mathcal{C}/P$ is $g\colon\ B \to P$, with unit $h\colon\ A \to B$. The covering g is necessarily a projection $S \times P \to P$ with S discrete; and now, since A is connected, h must be of the form kf where $k = s \times 1\colon\ P \to S \times P$ for some $s \in S$. Because h is the unit of the reflexion and $kgh = kf = h$, we have $kg = 1$; accordingly $S = \{s\},\ g = 1_P$, and $h = f$. Thus $f \in \mathcal{M}^{*\uparrow}$ by Section 2.12. Yet if U is the subset of P given by the points whose distance from $(1/2, 0)$ is less than $1/4$, the restriction $f^*(U) \to U$ of f is not in $\mathcal{E}$, since $f^*(U)$ is not connected; so $f \notin \mathcal{E}'$.

10.3. It remains to give what was the second of the counter-examples promised in Section 6.1: a factorization system $(\mathcal{E}, \mathcal{M})$ arising as $\Phi\mathcal{X}$ from an admissible reflexion, and having each $\mathcal{M}^*/B$ reflective in $\mathcal{C}/B$, but with $\mathcal{M}^*$ not closed under composition.

Taking any non-trivial group G, write $\widetilde{\mathcal{C}}$ for the category G-**Set** of sets with a G-action. Those objects whose G-action is trivial form a reflective full subcategory $\widetilde{\mathcal{X}}$, isomorphic to the category of sets; write $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{M}})$ for the corresponding reflective factorization system $\Phi\widetilde{\mathcal{X}}$. One verifies easily that the reflexion here is admissible, and moreover that $\widetilde{\mathcal{M}}^*$ consists of *all* the maps in $\widetilde{\mathcal{C}}$. Accordingly $\widetilde{\mathcal{E}}'$ reduces, by Proposition 6.7, to the isomorphisms alone, and $(\widetilde{\mathcal{E}}', \widetilde{\mathcal{M}}^*)$ is trivially a factorization system. So this is yet one more positive, although trivial, example of the question raised in Section 6.1; but we use it here only to construct the counter-example we seek. The one property of it that we make use of is the fact that $\widetilde{\mathcal{M}}^*/1$ strictly contains $\widetilde{\mathcal{M}}/1 = \widetilde{\mathcal{X}}$, so that there is some $C \notin \widetilde{\mathcal{X}}$ with $C \to 1$ in $\widetilde{\mathcal{M}}^*$; any other example with this property would do as well.

Write $\mathcal{C}$ for the category obtained from $\widetilde{\mathcal{C}}$ by adding a new terminal object T; thus $\mathcal{C}(A, B) = \widetilde{\mathcal{C}}(A, B)$ for $A, B \in \widetilde{\mathcal{C}}$, while $\mathcal{C}(A, T) = \{t_A\}$ and $\mathcal{C}(T, A)$ is empty for $A \in \widetilde{\mathcal{C}}$, and $\mathcal{C}(T, T) = 1_T$. Note that $\mathcal{C}$ admits finite limits, while $\widetilde{\mathcal{C}}$ is closed in $\mathcal{C}$ under pullbacks. Write $\mathcal{X}$ for the full subcategory of $\mathcal{C}$ given by T and the objects of $\widetilde{\mathcal{X}}$; it is reflective in $\mathcal{C}$, the reflexion of A being its reflexion in

$\widetilde{\mathcal{X}}$ when $A \in \widetilde{\mathcal{C}}$, and being T when $A = T$. This reflexion of $\mathcal{C}$ onto $\mathcal{X}$ is easily seen to be admissible; write $(\mathcal{E}, \mathcal{M})$ for the corresponding reflective factorization system $\Phi\mathcal{X}$.

Clearly a map $p\colon\ E \to B$ in $\widetilde{\mathcal{C}}$ is an effective descent map in $\mathcal{C}$ if and only if it is so in $\widetilde{\mathcal{C}}$. Of course the identity $1_T\colon\ T \to T$ is an effective descent map in $\mathcal{C}$. For $E \in \widetilde{\mathcal{C}}$, the unique $t_E\colon\ E \to T$ is not an effective descent map in $\mathcal{C}$; for t_E^* does not reflect isomorphisms, sending the non-invertible $1 \to T$ (where 1 is the terminal object of $\widetilde{\mathcal{C}}$) to the invertible $1 \times E \to E$.

It follows that $\mathcal{M}^*/B = \widetilde{\mathcal{M}}^*/B$ for $B \in \widetilde{\mathcal{C}}$, while $\mathcal{M}^*/T = \mathcal{M}/T = \mathcal{X}$. If we choose $C \in \widetilde{\mathcal{C}}$ as in the second paragraph of this section, the map $C \to 1$ is in $\widetilde{\mathcal{M}}^*$ and hence in $\mathcal{M}^*$, while $1 \to T$ is also in $\mathcal{M}^*$; but their composite $C \to T$ is not in $\mathcal{M}^*$ since $C \notin \mathcal{X}$.

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