CANADIAN MATHEMATICAL SOCIETY
SOCIÉTE MATHÉMATIQUE DU CANADA

Session IV - Categorical Logic and Computer Science/
Logique catégorique et informatique

R. Pare (Dalhousie)
Dinatural numbers

Sunday/Dimanche
11:05-11:45

WINTER MEETING
1993
RÉUNION D'HIVER

Hosted by

Carleton
UNIVERSITY

DINATURAL NUMBERS

ROBERT PARÉ
&
LEOPOLDO ROMÁN

DINATURAL TRANSFORMATIONS

(DUBUC / STREET)

$F, G : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{B}$

$t : F \xrightarrow{\cdot\cdot} G$

$t(A) : F(A,A) \longrightarrow G(A,A)$

$\forall f : A \longrightarrow A'$

$$\begin{tikzcd}
 & F(A,A) \arrow[r, "t(A)"] & G(A,A) \arrow[dr, "G(A{,}f)"] & \\
F(A',A) \arrow[ur, "F(f{,}A)"] \arrow[dr, "F(A'{,}f)"'] & & & G(A,A') \\
 & F(A',A') \arrow[r, "t(A')"'] & G(A',A') \arrow[ur, "G(f{,}A')"'] &
\end{tikzcd}$$

Ex:

- $tr : \mathrm{HOM}(V,V) \xrightarrow{\cdot\cdot} k$
- $ev : A^B \times B \xrightarrow{\cdot\cdot} A$

  $(f\ ,\ b) \longmapsto f(b)$
- $nt : F \xrightarrow{\cdot} G$

$$\mathrm{HOM}_{\underline{A}} : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{\mathrm{SET}}$$

$$t : \mathrm{HOM}_{\underline{A}} \xrightarrow{\ \ddot{}\ } \mathrm{HOM}_{\underline{A}}$$

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Hom}(A,A) \arrow[r] & \mathrm{Hom}(A,A) \arrow[dr] & \\
\mathrm{Hom}(A,B) \arrow[ur] \arrow[dr] & & & \mathrm{Hom}(B,A) \\
 & \mathrm{Hom}(B,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[ur] &
\end{tikzcd}$$

$\forall\, A \underset{g}{\overset{f}{\rightleftarrows}} B$ $\qquad \boxed{t(gf)g = g\,t(fg)}$

<u>**Ex:**</u> $t(f) = f^{(n)} = f \circ f \circ f \circ \cdots \circ f$

<u>QUESTION</u> (BAINBRIDGE/FREYD/SCEDROV/SCOTT)

FOR $\underline{A} = \underline{\mathrm{SET}}_0$, IS EVERY DINATURAL $t : \mathrm{HOM} \xrightarrow{\ \ddot{}\ } \mathrm{HOM}$ OF THE FORM $(\ )^{(m)}$ FOR SOME $m \in \mathbb{N}$?

COUNTER-EXAMPLES

- $A \xrightarrow{f} A \longmapsto f^{(2!\,3!\,4!\,\dots\,k!)}$

  $k = \# A$

- (JOHNSTONE) $f \mapsto f^{(n)}$ s.t. IDEMPOTENT

- (FREYD) $f \mapsto \begin{cases} f & \text{IF } \# \mathrm{Fix}(f) = m \\ 1_A & \text{o.w.} \end{cases}$

DINATS DON'T COMPOSE

Ex:

$$\begin{tikzcd}
 & k \arrow[r, "1_V"] & \mathrm{Hom}(V,V) \arrow[r, "tr"] & k \arrow[dr, equal] & \\
k \arrow[ur, equal] \arrow[dr, equal] & & & & k \\
 & k \arrow[r, "1_W"'] & \mathrm{Hom}(W,W) \arrow[r, "tr"'] & k \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(A,A) \arrow[rr, "t(A)"] & & G(A,A) \arrow[dr] \arrow[rr, "u(A)"] & & H(A,A) \arrow[dr] & \\
F(A,B) \arrow[ur] \arrow[dr] \arrow[r] & & G(A,B) \arrow[ur] \arrow[dr] & & G(B,A) & & H(B,A) \\
 & F(B,B) \arrow[rr, "t(B)"'] & & G(B,B) \arrow[ur] \arrow[rr, "u(B)"'] & & H(B,B) \arrow[ur] &
\end{tikzcd}$$

IF ONE IS N.T. – THEY DO.

IF IS P.B. OR P.O. – ALSO.

BARR DINATURAL TRANSFORMATIONS

$t : F \Longrightarrow G : \underline{A}^{op} \times \underline{A} \longrightarrow \underline{B}$ w. P.B.

$$\begin{tikzcd}
 & F(A,A) \arrow[r, "tA"] & G(A,A) \arrow[dr, "G(A,f)"] & \\
P_f \arrow[ur] \arrow[dr] & & \boxed{\forall f : A \to A'} & G(A,A') \\
 & F(A',A') \arrow[r, "tA'"'] & G(A',A') \arrow[ur, "G(f,A')"'] &
\end{tikzcd}$$

WHERE

$$\begin{tikzcd}
 & F(A,A) \arrow[dr, "F(A,f)"] & \\
P_f \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\boxed{PB}"] & & F(A,A') \\
 & F(A',A') \arrow[ur, "F(f,A')"'] &
\end{tikzcd}$$

- NAT $\Rightarrow$ BARR DINAT $\Rightarrow$ DINAT
- $ev : A^B \times B \longrightarrow A$ BDN
- BDNs COMPOSE
- F CONST IN 1 VAR $\Rightarrow$ BDN
- $t : \mathrm{HOM}_{\underline{A}} \Longrightarrow \mathrm{HOM}_{\underline{A}}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\psi"'] & A \arrow[d, "\psi"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
A \arrow[r, "t(f)"] \arrow[d, "\psi"'] & A \arrow[d, "\psi"] \\
B \arrow[r, "t(g)"'] & B
\end{tikzcd}$$

E.g.
$t(f) = f^{(n)}$.

PROP (BARR) EVERY BDN $t : \mathrm{Hom}_{\underline{\mathrm{SET}}} \Longrightarrow \mathrm{Hom}_{\underline{\mathrm{SET}}}$ IS OF THE FORM $t(f) = f^{(n)}$ FOR A UNIQUE $n \in \mathbb{N}$.

PROOF LET $n = t(s)(0)$ WHERE $s : \mathbb{N} \to \mathbb{N}$ IS SUCCESSOR.

$\forall f : A \to A$, $a \in A$

$$\begin{tikzcd}
 & \mathbb{N} \arrow[r, "s"] \arrow[d, dashed, "h"] & \mathbb{N} \arrow[d, dashed, "h"] \\
1 \arrow[ur, "0"] \arrow[r, "a"'] & A \arrow[r, "f"'] & A
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
 & \mathbb{N} \arrow[r, "t(s)"] \arrow[d, "h"] & \mathbb{N} \arrow[d, "h"] \\
1 \arrow[ur, "0"] \arrow[r, "a"'] & A \arrow[r, "t(f)"'] & A
\end{tikzcd}$$

$$h(n) = f^{(n)}(a) \Rightarrow t(f)(a) = h(t(s)(0))$$
$$= h(n) = f^{(n)}(a)$$
$$\Rightarrow t(f) = f^{(n)} \qquad \blacksquare$$

RMK: FREYD'S EXAMPLE IS NOT BDN

MINE & JOHNSTON'S IS – BASED ON ITERATION

RESULT IS MUCH MORE GENERAL

NATURAL NUMBERS OBJECT

$\underline{A}$ CAT W. $\times$, 1

A **NATURAL NUMBERS OBJECT** (LAWVERE)

IN $\underline{A}$ IS A DIAGRAM

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

S.T. $\forall$ $A \xrightarrow{g} B \xrightarrow{f} B$ $\exists!\, h$

$$\begin{tikzcd}
 & A\times N \arrow[r, "A\times s"] \arrow[d, dashed, "h"] & A\times N \arrow[d, dashed, "h"] \\
A\times 1 \arrow[ur, "A\times 0"] \arrow[r, "g"'] & B \arrow[r, "f"'] & B
\end{tikzcd}$$

(IN $\underline{\text{SET}}$, $h(a,n) = f^{(n)}(g(a))$.)

RMK: FOR $\underline{A}$ CARTESIAN CLOSED, SUFF TO STATE DEF FOR $A = 1$.

CAN BE EXPRESSED IN TERMS OF ADJOINTS

$$\begin{tikzcd}
\underline{A}^{\bullet\circlearrowleft} \arrow[d, shift left, "U"] \\
\underline{A} \arrow[u, shift left, "F"]
\end{tikzcd}
\qquad F(A) = (A\times N, A\times s)$$

$\underline{\text{NEC.}}$ $\boxed{F(B\times A) \cong B\times F(A)}$

GIVEN $n : 1 \longrightarrow N$ WE CAN DEFINE A BDN $(\ )^{(n)} : \mathrm{Hom}_{\underline{A}} \Longrightarrow \mathrm{Hom}_{\underline{A}}$ AS FOLLOWS:

$$A \xrightarrow{f} A \longmapsto A \xrightarrow{f^{(n)}} A$$

$$\begin{tikzcd} A \times 1 \arrow[r, "A\times 0"] \arrow[dr, "\cong"'] & A \times N \arrow[r, "A\times s"] \arrow[d, dashed, "h"] & A \times N \arrow[d, dashed, "h"] \\ & A \arrow[r, "f"'] & A \end{tikzcd}$$

$$A \xrightarrow{\cong} A \times 1 \xrightarrow{A\times n} A \times N \xrightarrow{h} A$$

$$f^{(n)} = h \cdot A \times n \cdot \rho^{-1}$$

THIS IS A BDN.

GIVEN A BDN $t : \mathrm{Hom} \Longrightarrow \mathrm{Hom}$ WE GET $n = (1 \xrightarrow{0} N \xrightarrow{t(s)} N)$.

STARTING WITH n WE GET THE SAME n BACK: $s^{(n)}(0) = n$.

OTHER WAY DOES NOT WORK IN GENERAL.

NEED $\boxed{t(A \times g) = A \times t(g)}$

<u>DEF</u>: IF t SATISFIES THIS WE CALL IT STRONG.

A BDN $t : \mathrm{Hom}_{\underline{A}} \Longrightarrow \mathrm{Hom}_{\underline{A}}$ IS SAME AS

$$
\begin{tikzcd}
\underline{A}^{\mathbb{T}} \arrow[rr, "T"] \arrow[dr, "U"'] & & \underline{A}^{\mathbb{T}} \arrow[dl, "U"] \\
& \underline{A} &
\end{tikzcd}
$$

IF $\underline{A}$ CARTESIAN CLOSED AND LEX THEN $\underline{A}^{\mathbb{T}}$ IS AN $\underline{A}$-ENRICHED CAT

$$[(A,\alpha),(B,\beta)] \rightarrowtail B^A \underset{\beta^A}{\overset{B^\alpha}{\rightrightarrows}} B^A$$

t IS STRONG BDN $\Longleftrightarrow$ T IS STRONG FUNCT.

THE CONDITION $F(A \times B) \cong A \times F(B)$ ON THE LEFT ADJOINT TO U FOR PARAMETERS IN NNO SAYS $F \dashv U$ IS A STRONG ADJUNCTION.

(CAN ALL BE FORMULATED WITHOUT CC+LEX)

$$
\begin{tikzcd}
A \arrow[r, "f"] & A \\
A \times B \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[r, "f \times g"] & A \times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}
\implies
\begin{tikzcd}
A \arrow[r, "tf"] & A \\
A \times B \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[r, "t(f \times g)"] & A \times B \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
B \arrow[r, "tg"'] & B
\end{tikzcd}
$$

SO $t(f \times g) = tf \times tg$.

THUS STRONG $\iff$ $t1_A = 1_A$

<u>Ex</u>: $\underline{A} = \underline{\mathrm{SET}}^{\cdot\circlearrowleft}$

$$
\begin{tikzcd}
(A,\alpha) \arrow[r, "f"] \arrow[d, "\varphi"'] & (A,\alpha) \arrow[d, "\varphi"] \\
(B,\beta) \arrow[r, "g"'] & (B,\beta)
\end{tikzcd}
\implies
\begin{tikzcd}
(A,\alpha) \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & (A,\alpha) \arrow[d, "\varphi"] \\
(B,\beta) \arrow[r, "\beta"'] & (B,\beta)
\end{tikzcd}
$$

IS A BDN BUT NOT STRONG!

<u>THM 1</u>: SUPPOSE $\underline{A}$ HAS A NNO. THEN EVERY STRONG BDN $t: \mathrm{HOM}_{\underline{A}} \Longrightarrow \mathrm{HOM}_{\underline{A}}$ IS OF THE FORM $t(f) = f^{(m)}$ FOR UNIQUE $m: 1 \longrightarrow N$.

PARAMETERS

GIVEN $m : L \longrightarrow N$ $\quad$ ($m = \langle m_\lambda \rangle_{\lambda \in L}$)

DEFINE $\left(A \xrightarrow{f} A\right) \longmapsto \left(A \times L \xrightarrow{f^{(m)}} A\right)$

$$\begin{tikzcd}
 & A \times N \arrow[r, "A \times s"] \arrow[d, dashed, "h"] & A \times N \arrow[d, dashed, "h"] \\
A \times 1 \arrow[ur, "A \times o"] \arrow[r, "\cong"'] & A \arrow[r, "f"'] & A
\end{tikzcd}$$

$$f^{(m)} = \left(A \times L \xrightarrow{A \times m} A \times N \xrightarrow{h} A\right)$$

THIS GIVES A BDN $(\ )^{(m)} : \mathrm{Hom}_{\underline{A}} \Longrightarrow \mathrm{Hom}_{\underline{A}}^{L}$

($\mathrm{Hom}_{\underline{A}}^{L}(A, B) := \mathrm{Hom}_{\underline{A}}(A \times L, B)$)

$t : \mathrm{Hom}_{\underline{A}} \Longrightarrow \mathrm{Hom}_{\underline{A}}^{L}$ IS **STRONG** IF

$\forall\ B \xrightarrow{g} B$

$$\begin{tikzcd}
A \times B \times L \arrow[r, "t(A \times g)"] \arrow[d, equal] & A \times B \arrow[d, equal] \\
A \times B \times L \arrow[r, "A \times tg"'] & A \times B
\end{tikzcd}$$

THM 1': THERE IS A NATURAL BIJECTION BETWEEN SBDNs $t : \mathrm{Hom}_{\underline{A}} \longrightarrow \mathrm{Hom}_{\underline{A}}^{L}$ AND $m : L \longrightarrow N$.

NOTE: THIS GIVES A UNIVERSAL PROPERTY FOR $\mathbb{N}$ AS A RIGHT ADJOINT! "THE ELEMENTS OF $\mathbb{N}$ ARE STRONG ITERATORS."

IF $\underline{A}$ DOES NOT HAVE A NNO WE STILL GET A FUNCTOR

$\mathcal{N} : \underline{A}^{op} \longrightarrow \underline{SET}$

$\mathcal{N}(L) = \{ t : Hom_{\underline{A}} \Longrightarrow Hom^L_{\underline{A}} \mid STRONG \}$

- $\mathcal{N}$ HAS $0 : 1 \longrightarrow \mathcal{N}$

  $A \xrightarrow{f} A \longmapsto A \times L \xrightarrow{p_1} A \qquad (f^{(0)} = 1_A)$

- $\mathcal{N}$ HAS $s : \mathcal{N} \longrightarrow \mathcal{N}$

  $A \xrightarrow{f} A \overset{st}{\longmapsto} A \times L \xrightarrow{t(f)} A \xrightarrow{f} A \qquad (f^{(sn)} = f^{(n)} \circ f)$

- $\mathcal{N}$ HAS $+ : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$

$(t+u)(f) = t(f)\, u(f)$

- $+$ IS ASSOCIATIVE
- $t+0=t$
- $t+su = s(t+u)$
- $+$ IS COMMUTATIVE

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & A \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r, "t(f)"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A \arrow[r, "t(f)"'] & A \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r, "t(f)"] \arrow[d, "u(f)"'] & A \arrow[d, "u(f)"] \\ A \arrow[r, "t(f)"'] & A \end{tikzcd}$$

$$\Rightarrow u(f)\,t(f) = t(f)\,u(f).$$

- $+$ IS NOT CANCELLATIVE!
- $+$ WITH PARAMETERS

$$A \xrightarrow{f} A \longmapsto A \times L \xrightarrow{(u(f),\,\pi_2)} A \times L \xrightarrow{t(f)} A.$$

- $\mathcal{N}$ HAS $\cdot : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$

$$(t \cdot u)(f) = t(u(f))$$

- $\cdot$ ASSOCIATIVE
- $0 \cdot t = 0$
- $1 \cdot t = t$
- $(t+u)\cdot v = t \cdot v + u \cdot v$

$\cdot$ NOT ALWAYS COMMUTATIVE!

$\mathcal{N}$ FOR $\underline{\mathrm{SET}}_{\mathrm{FIN}}$

FOR $\underline{\mathrm{SET}}_{\mathrm{FIN}}$ STRENGTH IS AUTOMATIC

$t(A\times g) = A\times t(g)$ ★

- ALWAYS FOR $A = 1$ TERMINAL OBJ.
- IF ★ HOLDS FOR A_i & $(\ )\times B$ PRESERVES Σ, THEN ★ HOLDS FOR ΣA_i.

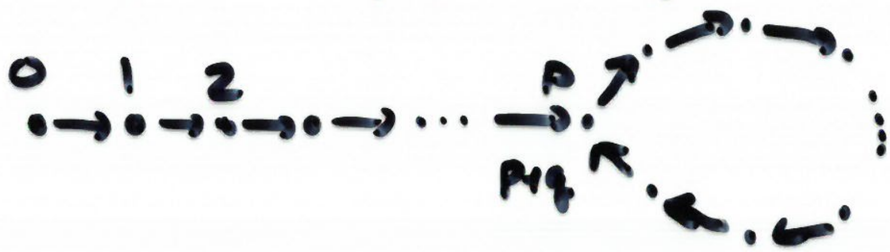
IF $(\ )\times B$ PRESERVES Σ, THEN $\mathcal{N}(\Sigma L_i) \cong \prod \mathcal{N}(L_i)$.

SUFFICIENT TO LOOK AT $\mathcal{N}(1)$ FOR $\underline{\mathrm{SET}}_{\mathrm{FIN}}$.

AS $\underline{\mathrm{SET}}_{\mathrm{FIN}}$ HAS NO NNO, WE WILL FIND A REPLACEMENT. WE WILL CALL THEM DINATURAL NUMBERS.

CONSIDER $(\mathbb{N}, 0, s)$ AS AN ALGEBRA OF TYPE $(0,1)$.

LET $A_{p,q} = \mathbb{N}/_{p = p+q}$, $p, q \in \mathbb{N}$, $q \neq 0$.



THERE IS AT MOST ONE HOMOMORPHISM

$$A_{p,q} \longrightarrow A_{p',q'}$$

AND THERE IS ONE $\iff p' \leq p$ & $q' \mid q$

HAVE A DIRECTED DIAGRAM OF ALGS.

$$\hat{\mathbb{N}} = \varprojlim_{p,q} A_{p,q}$$

NOTE THAT THE $A_{p,q}$ HAVE $+$ & $\cdot$

AND THE MORPHISMS PRESERVE THEM.

SO $\hat{\mathbb{N}}$ HAS $+$ & $\cdot$ (OBVIOUS ONES).

THM 2: THE BDN $t : \mathrm{HOM}_{\mathrm{SET}_{\mathrm{FIN}}} \Longrightarrow \mathrm{HOM}_{\mathrm{SET}_{\mathrm{FIN}}}$ CORRESPOND BIJECTIVELY TO ELEMENTS OF $\widehat{\mathbb{N}}$. THE $0, s, +, \cdot$ OF BDN's CORRESPOND TO $0, s, +, \cdot$ ON $\widehat{\mathbb{N}}$.

"PROOF" GIVEN $t : \mathrm{HOM} \Longrightarrow \mathrm{HOM}$

$$A_{p,q} \xrightarrow{s_{p,q}} A_{p,q} \overset{t}{\longmapsto} A_{p,q} \xrightarrow{t(s_{p,q})} A_{p,q}$$

DEF $\langle m_{p,q} \rangle_{p,q}$ BY $m_{p,q} = t(s_{p,q})(0)$.

THIS IS IN $\widehat{\mathbb{N}}$.

GIVEN $\langle m_{p,q} \rangle \in \widehat{\mathbb{N}}$ AND $f : A \to A$.

$\{1_A = f^0, f^1, f^2, \dots \}$ IS A FINITE CYCLIC MONOID SO $\cong A_{p,q}$ FOR SOME p, q.

DEFINE $t(f) = f^{m_{p,q}}$. $\square$

TO BETTER UNDERSTAND THE STRUCTURE OF $\hat{\mathbb{N}}$ WE CONSIDER AN INITIAL SUBDIAGRAM GIVEN BY $B_m = A_{m!,m!}$.

WE NOW HAVE

$$\hat{\mathbb{N}} \cdots \longrightarrow B_4 \longrightarrow B_3 \longrightarrow B_2 \longrightarrow B_1$$

THERE IS A COPY OF $\mathbb{N} \hookrightarrow \hat{\mathbb{N}}$ CORRESPONDING TO CONSTANT SEQUENCES. THESE ARE

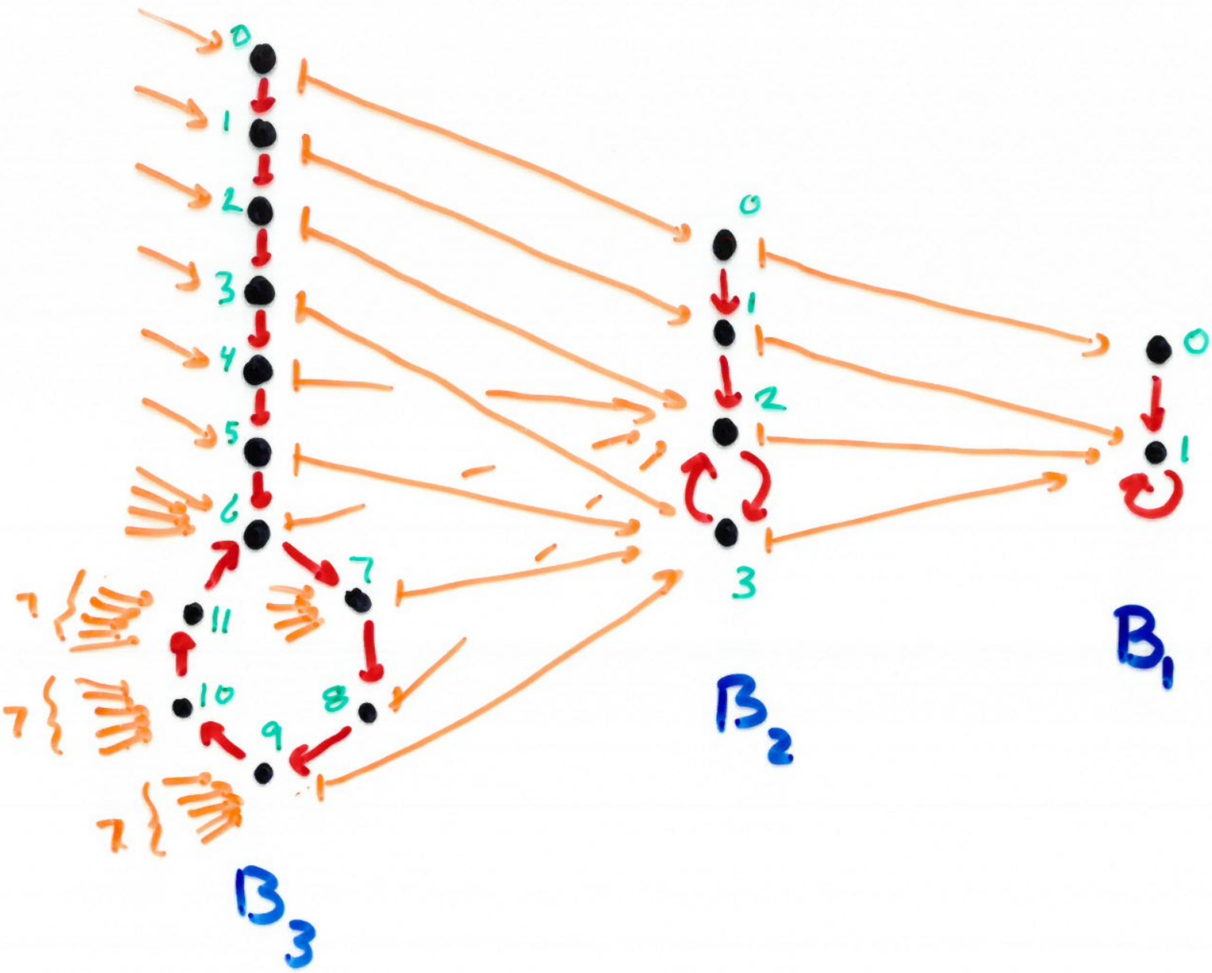
$0, S0 = 1, S1 = 2, \ldots.$

THERE IS ALSO AN ELEMENT

$$\omega = \langle [m!] \rangle_m$$

$$\omega + \omega = \omega \cdot \omega = \omega$$

BUT $\omega + 1 \neq \omega$.



$\omega = \langle 1, 2, 6, \ldots \rangle$

$\widehat{\mathbb{N}} \cong \mathbb{N} \cup \widehat{\mathbb{Z}}$, $\widehat{\mathbb{Z}} = \varprojlim_{n} \mathbb{Z}/(n)$

UNCOUNTABLE.