Journal of Pure and Applied Algebra 3 (1973) 21–42. © North-Holland Publishing Company

CONNECTED COMPONENTS AND COLIMITS *

Robert PARÉ
Department of Mathematics, Dalhousie University, Halifax, N.S., Canada

Communicated by S. MacLane
Received 10 December 1971

0. Introduction

A functor $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ is called *final* if for every $\boldsymbol{B}$ and every $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ the colimit of Γ exists whenever the colimit of $\Gamma\Phi$ exists and in this case they are isomorphic. It is well known (see for example [1]) that Φ is final if and only if the comma category (A, Φ) is nonempty and pathwise connected for every $A \in |\boldsymbol{A}|$.

In [12], a diagram $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ is said to have an *absolute colimit* if it has a colimit which is preserved by every functor $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ for every $\boldsymbol{B}$. A diagram Φ has an absolute colimit if and only if certain morphisms of $\boldsymbol{A}$ are connected in the comma category $(\Phi I, \Phi)$ for every $I \in |\boldsymbol{I}|$.

The similarity of these two properties and also of their characterizations leads us to look for a common generalization. We are naturally led to associate to the diagram $\Phi : \boldsymbol{I} \to \boldsymbol{A}$, the functor $\pi_0(-,\Phi) : \boldsymbol{A}^{\mathrm{op}} \to \boldsymbol{S}$ which associates to $A \in |\boldsymbol{A}|$ the set of pathwise connected components of the comma category (A, Φ).

The basic result, which we prove in Section 3, is the following: Given two diagrams $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ and $\Psi : \boldsymbol{J} \to \boldsymbol{A}$ then $\pi_0(-, \Phi) \simeq \pi_0(-, \Psi)$ if and only if for every $\boldsymbol{B}$ and every $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$, $\varinjlim \Gamma\Phi$ and $\varinjlim \Gamma\Psi$ exist simultaneously and when they do exist are isomorphic.

Thus all the "functorial" colimit properties of the diagram Φ are reflected in the functor $\pi_0(-, \Phi)$, whereas the "accidental" colimit properties of Φ are forgotten. If we take $\boldsymbol{J} = \boldsymbol{1}$, then we get the characterization for absolute colimits. If, instead, we take $\Psi = \boldsymbol{A}$ then we get the characterization for final functors.

This suggests that if we want to study colimit properties of a diagram we should study this associated functor rather than the diagram itself. So, after setting the stage in Section 1 with some preliminary material, we study in Section 2 the properties of the functor $\pi_0(-, \Phi)$. Section 3 relates these functors to colimits.

* This research was done while the author held a Killam Postdoctoral Fellowship at Dalhousie University, Halifax. A preliminary report on this work was presented at the Midwest Category Seminar in Zürich, August 1970.

One of the advantages of considering $\pi_0(-,\Phi)$ rather than Φ itself is that certain operations can be performed or conditions imposed on these functors which would be awkward on the diagrams. For example, natural transformations $\pi_0(-,\Phi) \to \pi_0(-,\Psi)$ are not always induced by morphisms of diagrams $\Phi \to \Psi$, and so we can take colimits of such natural transformations, an operation which would be difficult to perform on the diagrams themselves. The examples of Section 4 illustrate this.

It is a well known result, in disguise, that every functor $F: A^{op} \to S$ is of the form $\pi_0(-,\Phi)$ for some diagram Φ. Thus we can use the properties of colimits to obtain properties of set-valued functors. Also certain conditions or operations which are natural on diagrams are awkward on the functors. As Tierney has pointed out, the left Kan extension is more easily given in this context. He has used such methods to compute Kan extensions and to study their properties (see also [2]).

In practice, we are not interested in *all* functors $\Gamma : A \to B$ but often in a restricted class of functors such as coproduct preserving functors or finite colimit preserving functors. Thus, in Section 5 we restrict ourselves to functors $\Gamma : A \to B$ which preserve a given class of colimits in A. We get results analogous to those of Section 3. It is difficult to imagine how these characterizations could have been obtained without the functors $\pi_0(-,\Phi)$. In this relative sense, the absolute colimits are intuitively those whose existence is forced by the existence of the given class of colimits. This forcing is done in a functorial way.

In Section 6, we work out in detail some examples of the relative theory in the case of finite coproducts.

1. Preliminaries

All categories are assumed to be small unless they are clearly otherwise. Thus the categories denoted by $I, J, \ldots, A, B, \ldots$ will be small. The categories S and **Cat** of small sets and small categories, respectively, are large, as well as most categories constructed from these such as the functor category $S^{A^{op}}$ and the comma category $(\mathbf{Cat}, A)$.

The "hom" functor for a category A will be denoted by $A(-,-) : A^{op} \times A \to S$ or more frequently by $(-,-)$. Composition is written $A \xrightarrow{a} A' \xrightarrow{a'} A'' = a'a$, and the identity on A is written A.

Let A be a category and $A, A' \in |A|$. We say that A and A' are *pathwise connected* if there exists a finite number of objects of A, $A = A_0, A_1, \ldots, A_n = A'$, and for each $i = 1, 2, \ldots, n$ a morphism $A_{i-1} \to A_i$ *or* $A_i \to A_{i-1}$. This is an equivalence relation on the set of objects of A and the quotient set will be denoted by $\pi_0 A$, and called the set of (connected) components of A. π_0 extends to a functor $\mathbf{Cat} \to S$ which is left adjoint to the functor $dis : S \to \mathbf{Cat}$ which associates to a set the discrete category on that set. More explicitly, π_0 may be computed as the following coequalizer in **Cat** :

$$A^2 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} A \to \pi_0 A.$$

Given a functor $U : \boldsymbol{B} \to \boldsymbol{A}$, we say that it *has a left adjoint at* $A \in |\boldsymbol{A}|$ if the functor $(A, U-)$ is representable. If $(A, U-) \simeq (B, -)$, we say that *B is the value of the left adjoint to U at A.* Let $\boldsymbol{A}_0 \rightarrowtail \boldsymbol{A}$ be the full subcategory of $\boldsymbol{A}$ determined by those objects at which U has a left adjoint. The value of the left adjoint at an object is unique up to isomorphism, and once a representative is chosen for each object in $\boldsymbol{A}_0$, the left adjoint extends *uniquely* to a functor $F : \boldsymbol{A}_0 \to \boldsymbol{B}$. We will usually write $F : \boldsymbol{A} \to \boldsymbol{B}$ and specify that it is partially defined. In the sequel, when we speak of partially defined functors, we will mean a functor defined on a *full* subcategory of the domain category. Since the pullback of a full subcategory along any functor is still a full subcategory, composition of partially defined functors is easy (like the composition of partially defined functions in $\boldsymbol{S}$).

Remark. We could use the "profunctors" of Bénabou, but the extra generality is not needed and partially defined functors are more conceptual. For example, the colimit functor is usually thought of as a partially defined functor, left adjoint to the diagonal. Furthermore, the composition of partially defined functors is simple compared to the composition of profunctors.

Proposition 1.1. *Let* $\boldsymbol{C} \xrightarrow{V} \boldsymbol{B} \xrightarrow{U} \boldsymbol{A}$ *be functors and let F, G, H be the partially defined left adjoints of U, V and UV, respectively. If FA exists, then GFA exists if and only if HA exists and then they are isomorphic.*

Proof. $(A, UVC) \simeq (FA, VC)$; thus $(A, UV-) \simeq (FA, V-)$ and the result follows.

The statement "X exists if and only if Y exists, and when they do exist they are isomorphic" will be written $X \cong Y$. Thus the conclusion of Proposition 1.1 would be: FA exists $\Rightarrow GFA \cong HA$.

Proposition 1.2. *Let* $U : \boldsymbol{B} \to \boldsymbol{A}$ *and* $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ *be functors and let* $F : \boldsymbol{A} \to \boldsymbol{B}$ *be the partially defined left adjoint of U. Assume that* $F\Phi I$ *exists for every* $I \in |\boldsymbol{I}|$. *Then* $F\Phi$ *is a functor* $\boldsymbol{I} \to \boldsymbol{B}$ *and is the value of the left adjoint to* $U^{\boldsymbol{I}}$ *at* Φ.

Proof. The result follows from the following sequence of natural bijections:

$$\frac{\dfrac{\dfrac{\dfrac{F\Phi \to \Psi}{\boldsymbol{I}\text{-indexed compatible families } \langle F\Phi I \to \Psi I\rangle_I}}{\boldsymbol{I}\text{-indexed compatible families } \langle \Phi I \to U\Psi I\rangle_I}}{\Phi \to U\Psi}}{\Phi \to U^{\boldsymbol{I}}\Psi}$$

Proposition 1.3. *In the same situation as above, if* $\varinjlim \Phi$ *exists then* $F \varinjlim \Phi \cong \varinjlim F\Phi$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\Delta_B"] \arrow[d, "U"'] & B^I \arrow[d, "U^I"] \\ A \arrow[r, "\Delta_A"'] & A^I \end{tikzcd}$$

By hypothesis, Δ_A has a left adjoint at Φ and by Proposition 1.2, U^I has a left adjoint at Φ. The result follows from Proposition 1.1.

We will make much use of the so-called comma categories. Let $\Phi : I \to A$ and $\Psi : J \to A$ be functors. The *comma category* (Φ, Ψ) has as objects ordered triples $(I, \Phi I \xrightarrow{a} \Psi J, J)$, where $I \in |I|$, $J \in |J|$, $a \in A$. There will usually be no confusion if we denote this ordered triple by $\Phi I \xrightarrow{a} \Psi J$. A morphism from $\Phi I \xrightarrow{a} \Psi J$ to $\Phi I' \xrightarrow{a'} \Psi J'$ is an ordered pair of maps (i, j), $i \in I$, $j \in J$ such that

$$\begin{tikzcd} \Phi I \arrow[r, "a"] \arrow[d, "\Phi i"'] & \Psi J \arrow[d, "\Psi j"] \\ \Phi I' \arrow[r, "a'"] & \Psi J' \end{tikzcd}$$

commutes. (Φ, Ψ) comes equipped with two projections $\partial_0(\Phi, \Psi) : (\Phi, \Psi) \to I$ and $\partial_1(\Phi, \Psi) : (\Phi, \Psi) \to J$ defined in the obvious way.

Two special cases of particular interest are (A, Ψ) obtained by taking Φ to be the functor $\mathbf{1} \to A$ with value A, and $(\mathbf{Cat}, A)$ the category of diagrams in A obtained by taking Φ to be the identity on $\mathbf{Cat}$ and Ψ the functor $\mathbf{1} \to \mathbf{Cat}$ with value A.

In dealing with colimits, the following generalization of $(\mathbf{Cat}, A)$ involving the 2-structure of $\mathbf{Cat}$ is useful. Let $[\mathbf{Cat}, A]$ have the same objects as $(\mathbf{Cat}, A)$, i.e., diagrams in A. A morphism from $\Phi : I \to A$ to $\Psi : J \to A$ is a pair (Γ, γ) where $\Gamma : I \to J$ and $\gamma : \Phi \to \Psi\Gamma$,

$$\begin{tikzcd} I \arrow[rr, "\Gamma"] \arrow[dr, "\Phi"'] & \arrow[d, phantom, "\xRightarrow{\gamma}"] & J \arrow[dl, "\Psi"] \\ & A & \end{tikzcd}$$

2. The basic functor

For any functor $\Phi : I \to A$, a morphism $a : A \to A'$ of A induces a functor $(A', \Phi) \to (A, \Phi)$ by composition. Thus the comma category gives us a functor

$(-, \Phi)$: $A^{\mathrm{op}} \to \mathbf{Cat}$.

If we compose $(-, \Phi)$ with $\pi_0 : \mathbf{Cat} \to \boldsymbol{S}$, we get the functor

$$\pi_0(-, \Phi) : \boldsymbol{A}^{\mathrm{op}} \to \boldsymbol{S}.$$

If $A \xrightarrow{a} \Phi I$ is an object of (A, Φ), we will denote its equivalence class in $\pi_0(A, \Phi)$ by $[A \xrightarrow{a} \Phi I]$.

The basic object of our study is the functor $M : [\mathbf{Cat}, \boldsymbol{A}] \to \boldsymbol{S}^{\boldsymbol{A}^{\mathrm{op}}}$ which sends a diagram $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ to the functor $\pi_0(-, \Phi) : \boldsymbol{A}^{\mathrm{op}} \to \boldsymbol{S}$. If (Γ, γ) is a morphism $\Phi \to \Psi$ in $[\mathbf{Cat}, \boldsymbol{A}]$, then $M(\Gamma, \gamma) : \pi_0(-, \Phi) \to \pi_0(-, \Psi)$ is defined at A by

$$[A \xrightarrow{a} \Phi I] \mapsto [A \xrightarrow{a} \Phi I \xrightarrow{\gamma I} \Psi\Gamma I].$$

We will denote the restriction of M to $(\mathbf{Cat}, \boldsymbol{A})$ also by M.

The fundamental property of this construction is the following:

Theorem 2.1. *$\pi_0(-, \Phi) \simeq \varinjlim Y\Phi$, where $Y : \boldsymbol{A} \to \boldsymbol{S}^{\boldsymbol{A}^{\mathrm{op}}}$ is the Yoneda functor.*

Proof. The theorem says that $\pi_0(A, \Phi) \simeq \varinjlim_I (A, \Phi I)$, the colimit being taken in $\boldsymbol{S}$. A simple computation shows this to be true.

Corollary 2.2. *Let $A : \mathbf{1} \to \boldsymbol{A}$. Then $\pi_0(-, A) \simeq (-, A)$.*

Proposition 2.3. *Let $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ and let H be a partial left adjoint to Γ. Assume that HB exists. Then $\pi_0(B, \Gamma\Phi) \simeq \pi_0(HB, \Phi)$.*

Proof. $\pi_0(HB, \Phi) \simeq \varinjlim (HB, \Phi I) \simeq \varinjlim (B, \Gamma\Phi I) \simeq \pi_0(B, \Gamma\Phi)$.

Corollary 2.4. *If Γ has an everywhere defined left adjoint H, then $\pi_0(-, \Gamma\Phi) \simeq \pi_0(H-, \Phi)$.*

Proposition 2.5. *$\pi_0(-, \boldsymbol{A}) \simeq 1$, where $1 : \boldsymbol{A}^{\mathrm{op}} \to \boldsymbol{S}$ is the functor with constant value 1.*

Proof. The comma category $(A, \boldsymbol{A})$ has $A : A \to A$ as terminal object, and is therefore connected.

Proposition 2.6. *Let $\partial_0 : \boldsymbol{I}^2 \to \boldsymbol{I}$ be the functor which sends a morphism of $\boldsymbol{I}$ to its domain. Then $\pi_0(-, \partial_0) \simeq 1$.*

Proof. ∂_0 has a left adjoint, the diagonal functor $\Delta : \boldsymbol{I} \to \boldsymbol{I}^2$. Thus by Corollary 2.4, $\pi_0(-, \partial_0) \simeq \pi_0(\Delta-, \boldsymbol{I}) \simeq \pi_0(-, \boldsymbol{I})\cdot\Delta$ which is isomorphic to 1 by Proposition 2.5.

Let $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ be a functor. The Kan extension theorem shows that the functor $\boldsymbol{S}^{\Gamma^{\mathrm{op}}} : \boldsymbol{S}^{\boldsymbol{B}^{\mathrm{op}}} \to \boldsymbol{S}^{\boldsymbol{A}^{\mathrm{op}}}$ has a left adjoint (everywhere defined) which we shall denote by $\Gamma!$. One of the properties of $\Gamma!$ which we shall use often is that it makes the following square commute up to isomorphism:

$$\begin{tikzcd} A \arrow[r, "Y"] \arrow[d, "\Gamma"'] & S^{A^{op}} \arrow[d, "\Gamma!"] \\ B \arrow[r, "Y"'] & S^{B^{op}} \end{tikzcd}$$

Theorem 2.7. $\Gamma!\pi_0(-,\Phi) \simeq \pi_0(-,\Gamma\Phi)$.

Proof. $\pi_0(-,\Phi) \simeq \varinjlim Y\Phi$, thus $\Gamma!\,\pi_0(-,\Phi) \simeq \Gamma!\,\varinjlim Y\Phi \simeq \varinjlim \Gamma!\,Y\Phi \simeq \varinjlim Y\Gamma\Phi \simeq \pi_0(-,\Gamma\Phi)$.

Corollary 2.8. $\pi_0(-,\Phi) \simeq \Phi!(1)$.

Proof. $\pi_0(-,\Phi) \simeq \Phi!\,\pi_0(-,I) \simeq \Phi!(1)$.

Corollary 2.9. *The natural transformations $\pi_0(-,\Phi) \to F\Psi$ are in a natural bijection with the natural transformations $\pi_0(-,\Psi\Phi) \to F$.*

Proof. This is just a restatement of Theorem 2.7 using the adjointness between $\Gamma!$ and $S^{\Gamma^{op}}$.

From Corollary 2.9 we conclude that the natural transformations $\pi_0(-,\Phi) \to F$ are in natural bijection with the natural transformations $\pi_0(-,I) \to F\Phi$, but by Proposition 2.5, $\pi_0(-,I) \simeq 1$. Since the natural transformations $1 \to F\Phi$ correspond bijectively to elements of $\varprojlim F\Phi$, we conclude that n.t. $(\pi_0(-,\Phi), F) \simeq \varprojlim F\Phi$. If we take $\Phi = A : 1 \to A$, we get n.t. $((-,A), F) \simeq FA$, the Yoneda lemma.

Proposition 2.10. *Let $\Phi : I \to A$ and $\Psi : J \to A$ be functors and let $\partial_0(\Psi,\Phi) : (\Psi,\Phi) \to J$ be the canonical projection from the comma category. Then $\pi_0(\Psi -,\Phi) \simeq \pi_0(-,\partial_0(\Psi,\Phi))$.*

Proof. We only give the isomorphisms, the details being left to the reader. For $J \in |J|$,

$$\pi_0(\Psi J,\Phi) \rightleftarrows \pi_0(J,\partial_0(\Psi,\Phi)),$$

$$[\Psi J \xrightarrow{a} \Phi I] \mapsto [J \xrightarrow{1} J,\ \Psi J \xrightarrow{a} \Phi I],$$

$$[\Psi J \xrightarrow{a\cdot\Psi j} \Phi I] \leftarrow [J \xrightarrow{j} J',\ \Psi J' \xrightarrow{a} \Phi I].$$

Corollary 2.11. *If $P_1 : A \times I \to A$ is the projection onto A, then $\pi_0(-,P_1) \simeq \pi_0(I)$.*

Proof. Consider the functors $T_A : A \to \mathbf{1}$ and $T_I : I \to \mathbf{1}$. The comma category $(T_A, T_I) \simeq A \times I$ and $\partial_0(T_A, T_I) = P_1$. Thus by Proposition 2.10, $\pi_0(-, P_1) \simeq \pi_0(-, \partial_0(T_A, T_I)) \simeq \pi_0(T_A -, T_I) \simeq \pi_0(-, T_I)\cdot T_A$. But it is easily checked that $\pi_0(-, T_I) : \mathbf{1} \to S$ is just $\pi_0(I)$ and the result follows.

The following theorem can be found in [7] and [9].

Theorem 2.12. *The functor $M : (\mathbf{Cat}, A) \to S^{A^{\mathrm{op}}}$ has a right adjoint $\widetilde{M}$, and $M\widetilde{M} \simeq S^{A^{\mathrm{op}}}$.*

Proof. The value of $\widetilde{M}$ at $F \in S^{A^{\mathrm{op}}}$ is the corresponding fibered category over A with discrete fibers, i.e. the comma category (Y, F) with its projection $\partial_0(Y, F)$ onto A, where $Y : A \to S^{A^{\mathrm{op}}}$ is the Yoneda functor.

$M\widetilde{M}(F) = \pi_0(-, \partial_0(Y, F))$ which by Proposition 2.10 is isomorphic to $\pi_0(Y(-), F)$ which by Corollary 2.2 is isomorphic to n.t. $(Y(-), F)$ which by the Yoneda lemma is isomorphic to F. Thus $M\widetilde{M} \simeq S^{A^{\mathrm{op}}}$.

$\widetilde{M}M(\Phi)$ is the diagram $\partial_0 : (Y, \pi_0(-, \Phi)) \to A$. The objects of $(Y, \pi_0(-, \Phi))$ are equivalence classes $[A \to \Phi I] \in \pi_0(A, \Phi)$. A morphism $[A \to \Phi I] \to [A' \to \Phi I']$ is a morphism of A, $a : A \to A'$ such that $[A \xrightarrow{a} A' \to \Phi I'] = [A \to \Phi I]$. ∂_0 sends $[A \to \Phi I]$ to $A \in |A|$. The unit for the adjunction is given by the commutative triangle

$$\begin{tikzcd} I \arrow[rr, "H_\Phi"] \arrow[dr, "\Phi"'] & & (Y, \pi_0(-, \Phi)) \arrow[dl, "\partial_0"] \\ & A & \end{tikzcd}$$

where $H_\Phi : I \mapsto [\Phi I : \Phi I \to \Phi I]$. The adjointness is easily checked.

We see from this that every functor $F \in S^{A^{\mathrm{op}}}$ is of the form $\pi_0(-, \Phi)$ for some diagram Φ. Furthermore this is just the well-known fact that every functor is a co-limit of representables. Indeed,

$$F \simeq \pi_0(-, \partial_0(Y, F)) \simeq \varinjlim [(Y, F) \to A \to S^{A^{\mathrm{op}}}].$$

As Tierney points out, this gives us an efficient method for computing the values of the left Kan extension. Let $\Gamma : A \to B$ and let $F \in S^{A^{\mathrm{op}}}$. Then $F \simeq \pi_0(-, \Phi)$ for some Φ (the discrete fibration over A associated to F) and then $\Gamma!(F) \simeq \Gamma!(\pi_0(-, \Phi)) \simeq \pi_0(-, \Gamma\Phi)$. This idea has been developed extensively in [2].

We conclude this section with a result on categories of fractions which we shall use later. Let A be a category and Σ a subcategory with the same objects as A. Then $P : A \to A[\Sigma^{-1}]$ is defined by the fact that P sends the morphisms of Σ to isomorphisms in $A[\Sigma^{-1}]$ and is universal with this property (see [4]). $A[\Sigma^{-1}]$ has the same objects as A and has as morphisms $A \to A'$, equivalence classes of finite sequences of morphisms

$$A = A_0 \xrightarrow{a_1} A_1 \xleftarrow{s_2} A_2 \xrightarrow{a_3} \cdots \xleftarrow{s_m} A_m = A',$$

where $a_i \in A$ and $s_i \in \Sigma$. The equivalence relation is such that it insures that $A[\Sigma^{-1}]$ is a category, P is a functor, and for $s \in \Sigma$, $A \xleftarrow{s} A'$ considered as a morphism from A to $A' \xrightarrow{s} A$ considered as a morphism from A' to A.

Proposition 2.13. *With the above notation,* $\pi_0(-, P) \simeq 1$.

Proof. Consider $\pi_0(A, P)$. Then $[A : A \to A] \in \pi_0(A, P)$. Now assume that $A \to A'$ is a morphism of $A[\Sigma^{-1}]$ and therefore of the type

$$A = A_0 \xrightarrow{a_1} A_1 \xleftarrow{s_2} A_2 \xrightarrow{a_3} \cdots \xleftarrow{s_m} A_m = A'.$$

It is easily checked that this finite sequence of objects and maps is a path connecting the morphism in question with the identity on A. Thus $\pi_0(-, P) \simeq 1$.

3. The connection with colimits

Let $Y : A \to S^{A^{op}}$ be the Yoneda functor and let R be its partially defined left adjoint. R is defined on every representable and $A \xrightarrow{Y} S^{A^{op}} \xrightarrow{R} A = A$.

Proposition 3.1. *Let* $\Phi : I \to A$ *be a diagram in* A. *Then* $\varinjlim \Phi \simeq R\, \pi_0(-, \Phi)$.

Proof. $\pi_0(-, \Phi) \simeq \varinjlim Y\Phi$ by Theorem 2.1, therefore $R\ \pi_0(-, \Phi) \simeq R \varinjlim Y\Phi$. But R is defined on the values of $Y\Phi$, thus by Proposition 1.3, $R\,\pi_0(-, \Phi) \simeq \varinjlim RY\Phi \simeq \varinjlim \Phi$.

Recall that according to the convention made following Proposition 1.1, the statement of the preceding proposition means that Φ has a colimit if and only if $\pi_0(-, \Phi)$ has a reflection in A, and then they are isomorphic.

We can make Proposition 3.1 more precise in the following way: Let $\mu : \Phi \to A$ be a natural transformation (compatible family). This gives us a morphism in $[\mathbf{Cat}, A]$ from $\Phi : I \to A$ to $A : \mathbf{1} \to A$. Therefore we have a natural transformation $\pi_0(-, \mu) : \pi_0(-, \Phi) \to \pi_0(-, A) \simeq (-, A)$. Then $\mu : \Phi \to A$ is a colimit diagram if and only if $\pi_0(-, \mu) : \pi_0(-, \Phi) \to (-, A)$ is a reflection into the full subcategory of $S^{A^{op}}$ determined by the representables.

Conversely, if we have such a reflection $t : \pi_0(-, \Phi) \to (-, A)$, we get a natural transformation $\mu : \Phi \to A$ by defining $\mu I : \Phi I \to A$ to be the image of $[\Phi I : \Phi I \to \Phi I]$ under $t\Phi I : \pi_0(\Phi I, \Phi) \to (\Phi I, A)$.

Theorem 3.2. *Let $\Phi : \mathbf{I} \to \mathbf{A}$ and $\Psi : \mathbf{J} \to \mathbf{A}$ be diagrams in $\mathbf{A}$. Then $\varinjlim \Gamma\Phi \cong \varinjlim \Gamma\Psi$ for every category $\mathbf{B}$ and every functor $\Gamma : \mathbf{A} \to \mathbf{B}$ if and only if $\pi_0(-, \Phi) \simeq \pi_0(-,\Psi)$.*

Proof. ($\Leftarrow$).

$$\pi_0(-, \Phi) \simeq \pi_0(-, \Psi) \Rightarrow \Gamma!\, \pi_0(-, \Phi) \simeq \Gamma!\, \pi_0(-, \Psi) \Rightarrow \pi_0(-, \Gamma\Phi) \simeq \pi_0(-, \Gamma\Psi)$$
$$\Rightarrow R\, \pi_0(-, \Gamma\Phi) \cong R\, \pi_0(-, \Gamma\Psi) \Rightarrow \varinjlim \Gamma\Phi \cong \varinjlim \Gamma\Psi.$$

($\Rightarrow$). Let $\mathbf{B}$ be the full subcategory of $S^{A^{op}}$ determined by all the representables plus the two functors $\pi_0(-, \Phi)$ and $\pi_0(-, \Psi)$. Let $\Gamma : \mathbf{A} \to \mathbf{B}$ be the restriction of Y. Since by Theorem 2.1, $\varinjlim Y\Phi \simeq \pi_0(-, \Phi)$ in $S^{A^{op}}$, then $\varinjlim \Gamma\Phi \simeq \pi_0(-, \Phi)$ in $\mathbf{B}$. Also $\varinjlim \Gamma\Psi \simeq \pi_0(-, \Psi)$ in $\mathbf{B}$. Therefore, by the hypothesis of the theorem, $\pi_0(-,\Phi) \simeq \pi_0(-, \Psi)$.

According to the convention at the beginning of Section 1, the category $\mathbf{B}$ in the preceding theorem has to be small. This explains why $\mathbf{B}$ was chosen as it was when $S^{A^{op}}$ would have sufficed.

A diagram $\Phi : \mathbf{I} \to \mathbf{A}$ is defined in [12] to have an absolute colimit A if for every category $\mathbf{B}$ and every functor $\Gamma : \mathbf{A} \to \mathbf{B}$, $\varinjlim \Gamma\Phi = \Gamma A$.

Corollary 3.3. *A diagram $\Phi : \mathbf{I} \to \mathbf{A}$ has an absolute colimit if and only if $\pi_0(-, \Phi)$ is representable.*

Proof. If $A : \mathbf{1} \to \mathbf{A}$ is the trivial diagram with value A, then for every $\Gamma : \mathbf{A} \to \mathbf{B}$, $\varinjlim \Gamma A = \Gamma A$. Therefore Φ has absolute colimit A if and only if for every $\Gamma : \mathbf{A} \to \mathbf{B}$, $\varinjlim \Gamma\Phi \simeq \varinjlim \Gamma A$ which by Theorem 3.2 happens if and only if $\pi_0(-, \Phi) \simeq \pi_0(-, A) \simeq (-, A)$.

This can be made more precise in the following way. Let $\mu : \Phi \to A$ be a natural transformation. Then, as before, there is a natural transformation $\pi_0(-, \mu) : \pi_0(-,\Phi) \to (-, A)$. In this setting, $\mu : \Phi \to A$ is an absolute colimit diagram if and only if $\pi_0(-, \mu)$ is an isomorphism. For $\pi_0(-, \mu)$ to be an isomorphism it must have an inverse $\delta : (-, A) \to \pi_0(-, \Phi)$ which, by the Yoneda lemma, corresponds to an element of $\pi_0(A, \Phi)$. Let this element be $[d_0 : A \to \Phi I_0]$. The conditions expressing that $\pi_0(-, \mu)$ and δ are inverse are, (i) $A \xrightarrow{d_0} \Phi I_0 \xrightarrow{\mu I_0} A = A$, and (ii) for every $I \in |\mathbf{I}|$, $[\Phi I \xrightarrow{\mu I} A \xrightarrow{d_0} \Phi I_0] = [\Phi I : \Phi I \to \Phi I]$ in $\pi_0(\Phi I, \Phi)$. This is the characterization given in [12].

A functor $\Phi : \mathbf{I} \to \mathbf{A}$ is called *final* if for every functor $\Gamma : \mathbf{A} \to \mathbf{B}$, $\varinjlim \Gamma\Phi \cong \varinjlim \Gamma$.

The following result is well known; see [1] for example.

Corollary 3.4. *Φ is final if and only if $\pi_0(-, \Phi) \simeq 1$.*

Proof. By Proposition 2.5, $1 \simeq \pi_0(-, A)$ and the result then follows from Theorem 3.2.

Let $\Lambda : L \to S^{A^{\mathrm{op}}}$ be a diagram and assume that for every $L \in |L|$, $\Lambda L = \pi_0(-, \Phi_L)$ for some diagram $\Phi_L : I_L \to A$. This is always true by Theorem 2.12, but Φ_L need not be the canonical diagram $\tilde{M}\Lambda L$. The natural transformations $\Lambda l : \pi_0(-, \Phi_l) \to \pi_0(-, \Phi_{L'})$ are not necessarily induced by morphisms $\Phi_L \to \Phi_{L'}$ in $[\mathbf{Cat}, A]$. If for some $\Gamma : A \to B$, $\varinjlim \Gamma\Phi_L$ exists for every $L \in |L|$, then $\varinjlim \Gamma\Phi_L$ extends to a functor in L, namely $L \xrightarrow{\Lambda} S^{A^{\mathrm{op}}} \xrightarrow{\Gamma!} S^{B^{\mathrm{op}}} \xrightarrow{R} B$.

Theorem 3.5. *Let $\Lambda : L \to S^{A^{\mathrm{op}}}$ be as above. Then $\varinjlim \Gamma\Phi \cong \varinjlim_L (\varinjlim \Gamma\Phi_L)$ for all categories B and all functors $\Gamma : A \to B$ for which $\varinjlim \Gamma\Phi_L$ exists for every $L \in |L|$ if and only if $\pi_0(-, \Phi) \simeq \varinjlim_L \pi_0(-, \Phi_L) = \varinjlim \Lambda$.*

Proof. $\pi_0(-, \Phi) \simeq \varinjlim_L \pi_0(-, \Phi_L) \Rightarrow \Gamma!\, \pi_0(-, \Phi) \simeq \Gamma!\, \varinjlim_L \pi_0(-, \Phi_L) \simeq \varinjlim_L \Gamma!\, \pi_0(-, \Phi_L) \Rightarrow \pi_0(-, \Gamma\Phi) \simeq \varinjlim_L \pi_0(-, \Gamma\Phi_L)$ for every $\Gamma : A \to B$. If for every $L \in |L|$, $\varinjlim \Gamma\Phi_L$ exists, then $R\, \pi_0(-, \Gamma\Phi_L)$ also exists and we have $\varinjlim_L R\, \pi_0(-, \Gamma\Phi_L) \simeq R \varinjlim_L \pi_0(-, \Gamma\Phi_L) \simeq R\, \pi_0(-, \Gamma\Phi)$. Thus $\varinjlim \Gamma\Phi \simeq \varinjlim_L (\varinjlim \Gamma\Phi_L)$.

Conversely, let B be the full subcategory of $S^{A^{\mathrm{op}}}$ determined by the representables, the functors $\pi_0(-, \Phi_L)$ for every $L \in |L|$, $\pi_0(-, \Phi)$, and $\varinjlim_L \pi_0(-, \Phi_L)$. B is then a small category, and let $\Gamma : A \to B$ be the restriction of the Yoneda functor. Then $\varinjlim \Gamma\Phi_L$ exists and is $\pi_0(-, \Phi_L)$. From the hypothesis of the theorem we conclude that $\pi_0(-, \Phi) \simeq \varinjlim_L \pi_0(-, \Phi_L)$.

Corollary 3.6. *A natural transformation $\pi_0(-, \Phi) \to \pi_0(-, \Psi)$ is an epimorphism if and only if for every $\Gamma : A \to B$ for which $\varinjlim \Gamma\Phi$ and $\varinjlim \Gamma\Psi$ exist, the canonically induced map $\varinjlim \Gamma\Phi \to \varinjlim \Gamma\Psi$ is an epimorphism.*

Let $\Phi : I \to A$ be a diagram and $\mu : \Phi \to A$ a natural transformation. $\mu : \Phi \to A$ is a weak colimit diagram if for every $\mu' : \Phi \to A'$ there exists (not necessarily unique) $a : A \to A'$ such that $a\mu = \mu'$. We state the following proposition without proof. No further use will be made of it except in the two subsequent corollaries.

Proposition 3.7. *Every weak colimit of $\Gamma\Phi$ is a weak colimit of $\Gamma\Psi$ for all B and all $\Gamma : A \to B$ is and only if $\pi_0(-, \Psi)$ is a retract of $\pi_0(-, \Phi)$, i.e. if and only if there exist natural transformations $\pi_0(-, \Psi) \xrightarrow{t} \pi_0(-, \Phi) \xrightarrow{u} \pi_0(-, \Psi)$ such that $ut = \pi_0(-, \Psi)$.*

Corollary 3.8. *A diagram $\Phi : I \to A$ has an absolute weak colimit if and only if $\pi_0(-, \Phi)$ is a retract of a representable.*

If A has split idempotents, then a retract of a representable is itself representable. In this case, Φ has an absolute weak colimit if and only if it has an absolute colimit.

Let us say that a functor $\Phi : I \to A$ is *weakly final* if for every B and every $\Gamma : A \to B$, all weak colimits of $\Gamma\Phi$ are also weak colimits of Γ.

Corollary 3.9. *$\Phi : I \to A$ is weakly final if and only if there exists a natural transformation $1 \to \pi_0(-, \Phi)$.*

It is clear from the definitions that the composition of two final functors is again final and that the composition of two weakly final functors is weakly final.

If we have two functors $I' \xrightarrow{\Phi'} I \xrightarrow{\Phi} A$ such that $\Phi\Phi'$ is weakly final, then Φ is weakly final. Indeed, we have a canonical natural transformation $\pi_0(-, \Phi\Phi') \to \pi_0(-, \Phi)$ and thus the assertion is obvious in view of Corollary 3.9. However, if $\Phi\Phi'$ is final we cannot conclude that Φ is final, as we can see from the following example:

$$1 \xrightarrow{i_1} 1 + 1 \xrightarrow{\nabla} 1.$$

It is clear that if $\Phi : I \to A$ and $\Psi : J \to A$ are two diagrams, and if $\Lambda : I \to J$ is such that $\Psi\Lambda = \Phi$ and Λ is final then, as far as colimits are concerned, Φ and Ψ are equivalent diagrams. In the remainder of this section we show how our construction relates to this.

Let $\Phi : I \to A$ be a diagram. The unit $H_\Phi : I \to (Y, \pi_0(-, \Phi))$ of the adjointness $M \dashv \widetilde{M}$ was described in the proof of Theorem 2.12.

Theorem 3.10. *The functor H_Φ is final.*

Proof. H_Φ can be factored as follows: $H_\Phi = I \xrightarrow{Q} (A, \Phi) \xrightarrow{P} (Y, \pi_0(-, \Phi))$, where Q sends $I \in |I|$ to the object $\Phi I : \Phi I \to \Phi I$ in (A, Φ) and P sends $A \to \Phi I$ to $[A \to \Phi I]$ in $(Y, \pi_0(-, \Phi))$. Now Q has a left adjoint S, which sends $A \to \Phi I$ to $I \in |I|$, therefore by Corollary 2.4, $\pi_0(-, Q) \simeq \pi_0(S(-), I)$ which by Proposition 2.5 is isomorphic to 1. Thus Q is final.

It is readily verified that $P : (A, \Phi) \to (Y, \pi_0(-, \Phi))$ is equivalent to $P_\Sigma : (A, \Phi) \to (A, \Phi)[\Sigma^{-1}]$, where Σ consists of those morphisms of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & \Phi I \arrow[d, "\Phi\alpha"] & I \arrow[d, "\alpha"] \\ A \arrow[r] & \Phi I' & I' \end{tikzcd}$$

in (A, Φ). Then by Proposition (2.13), P is also final. The result follows from the fact that the composition of final functors is final.

We recall the following fact from the theory of categories of fractions (see [4, chapter I, 2.5]).

Proposition 3.11. *Let $X \underset{F}{\overset{U}{\leftrightarrows}} Z$ be an adjoint pair, $F \dashv U$, with unit $\eta: Z \to UF$ and counit $\epsilon : FU \to X$. Assume that ϵ is an isomorphism. Let Σ be a subcategory of Z such that* (i) *$\forall Z \in |Z|$, $\eta Z \in \Sigma$ and* (ii) *$\forall s \in \Sigma$, Fs is an isomorphism. Then Σ satisfies* *a calculus of left fractions and $Z[\Sigma^{-1}]$ is equivalent to X, the equivalence interchanging F and P_Σ.*

Let Σ be the subcategory of $(\mathbf{Cat}, A)$ consisting of those morphisms

$$\begin{tikzcd} I \arrow[rr, "\Lambda"] \arrow[dr, "\Phi"'] & & J \arrow[dl, "\Psi"] \\ & A & \end{tikzcd}$$

such that Λ is final. Then $M(\Lambda)$ is an isomorphism. Furthermore, by Theorem 3.10 the unit of the adjunction $M \dashv \tilde{M}$ is in Σ. Therefore, by Proposition 3.11, Σ satisfies a calculus of left fractions and $(\mathbf{Cat}, A)[\Sigma^{-1}]$ is equivalent to $S^{A^{\mathrm{op}}}$. Under this equivalence, a diagram Φ is sent to the functor $\pi_0(-, \Phi)$. So if we adopt the point of view that two diagrams Φ and Ψ with $\Phi = \Psi\Lambda$ should be identified if Λ is final, then we are automatically led to associate $\pi_0(-, \Phi)$ to the diagram Φ. This result shows that, although a natural transformation $\pi_0(-, \Phi) \to \pi_0(-, \Psi)$ is not necessarily induced by a morphism of diagrams in $(\mathbf{Cat}, A)$ or even in $[\mathbf{Cat}, A]$, there is a diagram Θ and morphisms $\Phi \to \Theta$ and $\Psi \to \Theta$ in $(\mathbf{Cat}, A)$ such that the morphism $\Psi \to \Theta$ is given by a final functor and the given natural transformation is equal to the induced natural transformation $\pi_0(-, \Phi) \to \pi_0(-, \Theta)$ followed by the inverse of the induced natural isomorphism $\pi_0(-, \Psi) \to \pi_0(-, \Theta)$.

4. Examples

Example 4.1. One might suggest, as a naive generalization of cofinal subsequence, that the following condition be imposed on a functor $\Phi : I \to A$: for every $A \in |A|$ there exist $I \in |I|$ and $a : A \to \Phi I$ a morphism of A. This is equivalent to saying that the canonical morphism $\pi_0(-, \Phi) \to 1$ is an epimorphism. By Corollary 3.6, this is equivalent to the condition that for every $\Gamma : A \to B$ for which $\varinjlim \Gamma\Phi$ and $\varinjlim \Gamma$ exist, the induced morphism $\varinjlim \Gamma\Phi \to \varinjlim \Gamma$ be an epimorphism.

Example 4.2. Let $P : A \times I \to A$ be the projection onto the first factor. By Corollary 2.11, $\pi_0(-, P) \simeq \pi_0(I)$ which is isomorphic to $\pi_0(I)\cdot 1$ (the coproduct of $\pi_0(I)$ copies of 1). Therefore $\pi_0(-, P) \simeq \pi_0(I)\cdot\pi_0(-, A)$, and by Theorem 3.5 this implies that for every $\Gamma : A \to B$ for which $\varinjlim \Gamma$ exists, $\varinjlim \Gamma P \simeq \pi_0(I)\cdot\varinjlim \Gamma$.

Example 4.3. Let $\Phi : |I| \to I$ be the inclusion and let $D_0, D_1 : |I^2| \to |I|$ be the functors which send a morphism to its domain and codomain, respectively.

$\pi_0(-, \Phi) = \coprod_I (-, I)$, thus $\pi_0(I, \Phi)$ is the set of all morphisms of I with domain I. $\pi_0(-, \Phi D_0) = \coprod_{I \to \cdot} (-, I)$ thus $\pi_0(I, \Phi D_0)$ is the set of all composable pairs of morphisms of I such that the domain of the first is I.

We have two natural transformations $\pi_0(-, \Phi D_0) \rightrightarrows \pi_0(-, \Phi)$. One sends the composable pair $I \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ to the morphism $I \xrightarrow{\alpha} \cdot$ and is induced by the morphism of diagrams

$$\begin{tikzcd}
{|\mathbf{I}^2|} \arrow[rr, "D_0"] \arrow[dr, "\Phi D_0"'] & & {|\mathbf{I}|} \arrow[dl, "\Phi"] \\
& \mathbf{I} &
\end{tikzcd}$$

The other sends the composable pair $I \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ to the morphism $I \xrightarrow{\beta\alpha} \cdot$ and is induced by the morphism of diagrams

$$\begin{tikzcd}
{|\mathbf{I}^2|} \arrow[rr, "D_1"] \arrow[dr, "\Phi D_0"'] & \arrow[d, phantom, "\xRightarrow{t}"] & {|\mathbf{I}|} \arrow[dl, "\Phi"] \\
& \mathbf{I} &
\end{tikzcd}$$

where $t : \Phi D_0 \to \Phi D_1$ is the canonical natural transformation.

The coequalizer of these two natural transformations is easily seen to be $1 \simeq \pi_0(-, \mathbf{I})$. Therefore, by Theorem 3.5, for any $\Gamma : \mathbf{I} \to \mathbf{B}$ for which $\coprod_I \Gamma(I)$ and $\coprod_{I\to\cdot} \Gamma(I)$ exist, $\varinjlim \Gamma$ is isomorphic to the coequalizer of the two induced maps

$$\coprod_{I\to\cdot} \Gamma(I) \rightrightarrows \coprod_I \Gamma(I).$$

This is the well-known construction for colimits. This construction is absolute in the sense that any functor which preserves the two coproducts in question will preserve the colimit if and only if it preserves the coequalizer.

Example 4.4. If we assume that $\mathbf{I}$ has binary products, we obtain the following variant of Example 4.3. Let $\Phi : |\mathbf{I}| \to \mathbf{I}$ be the inclusion as before. Let $\Pi : |\mathbf{I}| \times |\mathbf{I}| \to \mathbf{I}$ be the functor which sends a pair of objects to their product in $\mathbf{I}$.

$\pi_0(I, \Pi)$ is the set of pairs of maps with domain I, i.e., $\pi_0(I, \Pi) \simeq \pi_0(I, \Phi) \times \pi_0(I, \Phi)$. We have the two projections

$$\pi_0(-, \Phi) \times \pi_0(-, \Phi) \rightrightarrows \pi_0(-, \Phi)$$

whose coequalizer is quite obviously $1 \simeq \pi_0(-, \mathbf{I})$.

Therefore, for any $\Gamma : \mathbf{I} \to \mathbf{B}$ for which $\coprod_I \Gamma(I)$ and $\coprod_{I,I'} \Gamma(I \times I')$ exist, $\varinjlim \Gamma$ is isomorphic to the coequalizer of

$$\coprod_{I,I'} \Gamma(I \times I') \rightrightarrows \coprod_I \Gamma(I).$$

This construction is also absolute in the above sense.

The dual situation shows that the sheaf axiom says that certain limits are preserved (see [6]).

Example 4.5. Another variant of Example 4.3 is the following: Let us say that a category $\boldsymbol{I}$ is *finitely generated* if it has finitely many objects and if there is a finite set X of morphisms of $\boldsymbol{I}$ such that any morphism of $\boldsymbol{I}$ is a composition of morphisms from X.

Then if we let Φ be the same as before but replace Ψ by its restriction to X, the argument of Example 4.3 goes through and we see that the colimit of any diagram $\Gamma : \boldsymbol{I} \to \boldsymbol{B}$ can be computed as the coequalizer of

$$\coprod_{I \to \cdot \in X} \Gamma(I) \rightrightarrows \coprod_{I} \Gamma(I).$$

Both coproducts are finite, so if a category $\boldsymbol{B}$ has all finite colimits, then it has all finitely generated colimits. Also, if a functor preserves these finite colimits, then it preserves finitely generated colimits.

Any diagram is a filtered colimit of its finitely generated subdiagrams in $(\mathbf{Cat}, \boldsymbol{A})$, therefore we conclude that all colimits can be constructed from finite colimits and filtered colimits.

5. The relative theory

In practice, we are not concerned with all functors $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ but often with a restricted class of functors. In this section, we consider those functors $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ which send a given set of cocones in $\boldsymbol{A}$ to colimit diagrams in $\boldsymbol{B}$. Thus we want to know which diagrams are essentially the same as far as colimits are concerned if we force certain cocones to be colimit diagrams. If the cocones we are given are already colimits, then the absolute colimits are intuitively those colimits that can be "functorially constructed" from the given ones.

A *cocone* in $\boldsymbol{A}$ is a natural transformation $\mu : \Theta \to A$ where Θ is a diagram in $\boldsymbol{A}$ and A is an object of $\boldsymbol{A}$. Let C be a set of cocones in $\boldsymbol{A}$. A functor $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ will be called a *C-functor* if it transforms the cocones of C to colimits in $\boldsymbol{B}$. In general, a prefix "C" will mean that we are considering only C-functors rather than all functors, e.g., C-final, C-absolute, etc.

Let $\boldsymbol{S}_C^{\boldsymbol{A}^{op}}$ denote the full subcategory of $\boldsymbol{S}^{\boldsymbol{A}^{op}}$ consisting of those functors $\boldsymbol{A}^{op} \to \boldsymbol{S}$ which convert the cocones of C to limits in $\boldsymbol{S}$. Let $i_C : \boldsymbol{S}_C^{\boldsymbol{A}^{op}} \to \boldsymbol{S}^{\boldsymbol{A}^{op}}$ be the inclusion. Gabriel and Ulmer [3], Popescu [13], Freyd and Kelly, and others have shown that i_C has a left adjoint R_C.

Some examples of reflective subcategories of functor categories obtained in this way are the following:

(a). Sheaves on a Grothendieck topology.

(b). Algebras over a theory in the sense of Lawvere [8].

(c). **Cat** $\rightarrowtail S^{\Delta^{op}}$ where Δ is the category of finite ordinals with order preserving maps.

(c′). **Monoids** $\rightarrowtail$ **Cat** $\rightarrowtail S^{\Delta^{op}}$.

(c″). $S \rightarrowtail$ **Cat** $\rightarrowtail S^{\Delta^{op}}$.

(d). All functors $A^{op} \to S$ which invert a given set of maps of A (F converts $A \overset{A}{\underset{A}{\rightrightarrows}} A \overset{x}{\to} A'$ to an equalizer $\Leftrightarrow$ Fx is invertible).

(e). All functors $A^{op} \to S$ which identify certain maps in the same hom sets (F converts $A \overset{x}{\underset{y}{\rightrightarrows}} A' \overset{A'}{\to} A'$ to an equalizer $\Leftrightarrow$ $Fx = Fy$). Thus we can choose all functors which make certain diagrams commute.

More details on this subject may be found in the recent paper of Gabriel and Ulmer [3].

Lemma 5.1. *The functor*

$$A \xrightarrow{Y} S^{A^{op}} \xrightarrow{R_{\mathcal{C}}} S_{\mathcal{C}}^{A^{op}}$$

sends the cocones of $\mathcal{C}$ *to colimit diagrams in* $S_{\mathcal{C}}^{A^{op}}$.

Proof. Let $\mu : \Theta \to A$ be a cocone in $\mathcal{C}$ where $\Theta : I \to A$ is a diagram in A. We want to show that

$$R_{\mathcal{C}} Y\mu : R_{\mathcal{C}} Y\Theta \to R_{\mathcal{C}} YA$$

is a colimit diagram in $S_{\mathcal{C}}^{A^{op}}$. Let $\Lambda \in S_{\mathcal{C}}^{A^{op}}$, then the result follows from the following sequence of natural bijections:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{cocones } \lambda : R_{\mathcal{C}} Y\Theta \to \Lambda \text{ in } S_{\mathcal{C}}^{A^{op}}}{I\text{-indexed compatible families } \langle \lambda_I : R_{\mathcal{C}} Y\Theta I \to \Lambda \rangle \text{ in } S_{\mathcal{C}}^{A^{op}}}}{I\text{-indexed compatible families } \langle \bar{\lambda}_I : Y\Theta I \to \Lambda \rangle \text{ in } S^{A^{op}}}}{I\text{-indexed compatible families } \langle l_I \in \Lambda\Theta I \rangle \text{ in } S}}{l \in \varprojlim \Lambda\Theta \text{ in } S}}{l \in \Lambda A \text{ in } S}}{YA \to \Lambda \text{ in } S^{A^{op}}}}{R_{\mathcal{C}} YA \to \Lambda \text{ in } S_{\mathcal{C}}^{A^{op}}.}$$

Theorem 5.2. *Let* $\Phi : I \to A$ *and* $\Psi : J \to A$ *be diagrams in* A. *Then* $\varinjlim \Gamma\Phi \cong \varinjlim \Gamma\Psi$ *for every category* B *and every* $\mathcal{C}$*-functor* $\Gamma : A \to B$ *if and only if* $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq R_{\mathcal{C}}\pi_0(-, \Psi)$.

Proof. Assume that $R_C\pi_0(-,\Phi) \simeq R_C\pi_0(-,\Psi)$ and let $\Gamma : A \to B$ be a C-functor.

First we show that there exists a functor V making the following diagram commute:

$$\begin{tikzcd} S_C^{A^{op}} \arrow[r, tail, "i_C"] & S^{A^{op}} \\ B \arrow[u, "V"] \arrow[r, tail, "Y"] & S^{B^{op}} \arrow[u, "S^{\Gamma^{op}}"'] \end{tikzcd}$$

Since $S_C^{A^{op}}$ is a full subcategory of $S^{A^{op}}$, it is sufficient to see that the values of $S^{\Gamma^{op}}\cdot Y$ lie in $S_C^{A^{op}}$.

$S^{\Gamma^{op}}\cdot Y(B) = (\Gamma-, B)$ and since Γ sends cocones in C to colimit diagrams in B and $(-, B)$ transforms colimits in B to limits in S, $(\Gamma-, B)$ is in $S_C^{A^{op}}$, and the existence of V follows.

Let $R : S^{B^{op}} \to B$, $F : S^{A^{op}} \to B$, and $G : S_C^{A^{op}} \to B$ be the partially defined left adjoints of Y, $S^{\Gamma^{op}}\cdot Y = i_C V$, and V, respectively. By Proposition 3.1, $\varinjlim \Gamma\Phi \cong R\,\pi_0(-,\Gamma\Phi)$ which is $\cong R\,\Gamma!\,\pi_0(-,\Phi)$ by Theorem 2.7. By Proposition 1.1, $R\,\Gamma!\,\pi_0(-,\Phi) \cong F\pi_0(-,\Phi)$ which by using Proposition 1.1 again is $\cong GR_C\pi_0(-,\Phi)$. Similarly, $\varinjlim \Gamma\Psi \cong GR_C\pi_0(-,\Psi)$. Therefore $\varinjlim \Gamma\Phi \cong \varinjlim \Gamma\Psi$.

Conversely, assume that $\varinjlim \Gamma\Phi \cong \varinjlim \Gamma\Psi$ for all C-functors Γ. Lemma 5.1 says that, except for the fact that $S_C^{A^{op}}$ is not necessarily small, $A \xrightarrow{Y} S^{A^{op}} \xrightarrow{R_C} S_C^{A^{op}}$ is a C-functor. Let B be the full subcategory of $S_C^{A^{op}}$ determined by the objects $R_C\pi_0(-,\Phi)$, $R_C\pi_0(-,\Psi)$, and $B_C(-,A)$ for every $A \in |A|$, and let $\Gamma : A \to B$ be $R_C Y$ with restricted codomain. Then Γ is a C-functor.

$\varinjlim R_C Y\Phi \simeq R_C \varinjlim Y\Phi \simeq R_C\pi_0(-,\Phi)$ by Theorem 2.1, and similarly $\varinjlim R_C Y\Psi \simeq R_C\pi_0(-,\Psi)$. Therefore, $\varinjlim \Gamma\Phi \simeq R_C\pi_0(-,\Phi)$ and $\varinjlim \Gamma\Psi \simeq R_C\pi_0(-,\Psi)$. Since Γ is a C-functor, we conclude that $R_C\pi_0(-,\Phi) \simeq R_C\pi_0(-,\Psi)$.

Let us say that a diagram $\Phi : I \to A$ has a *C-absolute colimit* if there is a cocone in A, $\mu : \Phi \to A$ such that for every C-functor $\Gamma : A \to B$, $\Gamma\mu : \Gamma\Phi \to \Gamma A$ is a colimit diagram in B.

This terminology may be slightly misleading if C does not consist of colimit diagrams. In that case, a C-absolute colimit is not necessarily a colimit in A. If, however, C consists entirely of colimit diagrams (this is usually the case), then the identity $A : A \to A$ is a C-functor and therefore a C-absolute colimit is indeed a colimit in A.

Corollary 5.3. *A diagram $\Phi : I \to A$ has a C-absolute colimit if and only if $R_C\pi_0(-,\Phi) \simeq R_C(-,A)$ for some $A \in |A|$.*

Proof. Let $A : 1 \to A$ be the diagram with value A. For every C-functor Γ, $\varinjlim \Gamma A = \Gamma A$. Therefore Φ has a C-absolute colimite A if and only if $\varinjlim \Gamma\Phi \simeq \varinjlim \Gamma A$ for every

$\mathcal{C}$-functor Γ. Theorem 5.2 says that this is equivalent to the condition $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq R_{\mathcal{C}}(-, A)$.

More precisely, if $\mu : \Phi \to A$ is a cocone then it is a $\mathcal{C}$-absolute colimit if and only if $R_{\mathcal{C}}\pi_0(-, \mu) : R_{\mathcal{C}}\pi_0(-, \Phi) \to R_{\mathcal{C}}(-, A)$ is an isomorphism.

If $\mathcal{C}$ consists entirely of colimit diagrams, then $(-, A)$ sends these colimits to limits in $\boldsymbol{S}$, so $R_{\mathcal{C}}(-, A) \simeq (-, A)$. In this case, Corollary 5.3 can be restated as follows.

Corollary 5.3′. *Φ has a $\mathcal{C}$-absolute colimit if and only if $R_{\mathcal{C}}\pi_0(-, \Phi)$ is representable.*

A functor $\Phi : \boldsymbol{I} \to \boldsymbol{A}$ is said to be *$\mathcal{C}$-final* if for every $\mathcal{C}$-functor $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$, $\varinjlim \Gamma\Phi \simeq \varinjlim \Gamma$.

Corollary 5.4. *Φ is $\mathcal{C}$-final if and only if $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq 1$.*

Proof. Clearly, $1 \simeq R_{\mathcal{C}}1$, and by Proposition 2.5, $1 \simeq \pi_0(-, A)$. Thus $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq 1$ if and only if $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq R_{\mathcal{C}}\pi_0(-, A)$ and the result is immediate by Theorem 5.2.

Theorem 5.5. *Let $\Lambda : \boldsymbol{L} \to \boldsymbol{S}^{A^{\mathrm{op}}}$ be as in Theorem 3.5. Then $\varinjlim \Gamma\Phi \cong \varinjlim_L (\varinjlim \Gamma\Phi_L)$ for all $\mathcal{C}$-functors $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ for which $\varinjlim \Gamma\Phi_L$ exists for every $L \in |\boldsymbol{L}|$ if and only if $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq R_{\mathcal{C}} \varinjlim_L \pi_0(-, \Phi_L) \simeq \varinjlim_L R_{\mathcal{C}}\pi_0(-, \Phi_L)$.*

Proof. $\varinjlim_L \pi_0(-, \Phi_L) \simeq \pi_0(-, \Psi)$ for some Ψ (Theorem 2.12), and by Theorem 3.15, $\varinjlim \Gamma\Psi \cong \varinjlim_L (\varinjlim \Gamma\Phi_L)$ for *all* functors $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$. The result then follows from Theorem 5.2.

Corollary 5.6. *A natural transformation $R_{\mathcal{C}}\pi_0(-, \Phi) \to R_{\mathcal{C}}\pi_0(-, \Psi)$ is an epimorphism in $\boldsymbol{S}^{A^{\mathrm{op}}}_{\mathcal{C}}$ if and only if for every $\mathcal{C}$-functor $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ for which $\varinjlim \Gamma\Phi$ and $\varinjlim \Gamma\Psi$ exist, the canonical map $\varinjlim \Gamma\Phi \to \varinjlim \Gamma\Psi$ is an epimorphism in $\boldsymbol{B}$.*

The preceding results suggest that we consider $\varinjlim \Gamma\Phi$ for all $\mathcal{C}$-functors as stalks for the functor $\pi_0(-, \Phi)$. This point of view has been used in certain computations involving $R_{\mathcal{C}}$ which do not appear in this paper. However, we have not systematically investigated this aspect of the theory presented here.

We end this section with some results on weak colimits analogous to those at the end of Section 3.

Proposition 5.7. *Every weak colimit of $\Gamma\Phi$ is a weak colimit of $\Gamma\Psi$ for all $\mathcal{C}$-functors $\Gamma : \boldsymbol{A} \to \boldsymbol{B}$ if and only if $R_{\mathcal{C}}\pi_0(-, \Psi)$ is a retract of $R_{\mathcal{C}}\pi_0(-, \Phi)$.*

Corollary 5.8. *$\Phi : \boldsymbol{I} \to \boldsymbol{A}$ is weakly $\mathcal{C}$-final if and only if there exists a natural transformation $1 \to R_{\mathcal{C}}\pi_0(-, \Phi)$.*

Corollary 5.9. *Assume that $\mathcal{C}$ consists of colimit diagrams. Then $\Phi : I \to A$ has a $\mathcal{C}$-absolute weak colimit if and only if $R_{\mathcal{C}}\pi_0(-, \Phi)$ is a retract of a representable.*

6. Examples

In general, the reflector $R_{\mathcal{C}}$ is difficult to describe explicitly. In this section, we develop some theory which gives us sufficient conditions and then show that in the case where $\mathcal{C}$ is obtained by considering all finite coproducts, these conditions are also necessary. We investigate this situation in more detail.

Let I be a category and $\mathcal{D}$ a set of diagrams in I. We want to add the colimits of the diagrams in $\mathcal{D}$ to I in a free manner. This we do as follows:

Let $\widetilde{I}$ be the full subcategory of $S^{I^{\mathrm{op}}}$ whose objects are the representables and the functors of the form $\pi_0(-, D)$ for every $D \in \mathcal{D}$. The Yoneda functor gives us a full embedding $H : I \to \widetilde{I}$.

H has the following characteristic properties:

(i). For every $D \in \mathcal{D}$, HD has a colimit in $\widetilde{I}$.

(ii). Every functor $\Phi : I \to A$ such that ΦD has a colimit in A for every $D \in \mathcal{D}$, extends to a functor $\widetilde{\Phi} : \widetilde{I} \to A$ such that $\widetilde{\Phi}$ preserves the colimit of HD for every $D \in \mathcal{D}$ and $\widetilde{\Phi}H \simeq \Phi$.

(iii). If $\Theta : \widetilde{I} \to A$ is any functor, then a natural transformation $\phi : \Phi \to \Theta H$ extends *uniquely* to a natural transformation $\widetilde{\phi} : \widetilde{\Phi} \to \Theta$ such that $\widetilde{\phi}H = \phi$.

In particular, condition (iii) implies that the $\widetilde{\Phi}$ of (ii) is unique up to isomorphism.

Theorem 6.1. *Let $\mathcal{C}$ be a set of cocones in A and let $\Phi : I \to A$ be such that for every $D \in \mathcal{D}$, ΦD has a colimit and the cocone thus obtained is in $\mathcal{C}$. Then $R_{\mathcal{C}}\pi_0(-, \Phi) \simeq R_{\mathcal{C}}\pi_0(-, \widetilde{\Phi})$.*

Proof. Let $\Gamma : A \to B$ be a $\mathcal{C}$-functor. Then $\Gamma\widetilde{\Phi}$ preserves the colimit of HD for every $D \in \mathcal{D}$, and $\Gamma\widetilde{\Phi}H \simeq \Gamma\Phi$. Since the extension of $\Gamma\Phi$ to $\widetilde{I}$ is unique up to isomorphism, $\widetilde{\Gamma\Phi} \simeq \Gamma\widetilde{\Phi}$. If $B \in |\mathbf{B}|$, then we have the following sequence of natural bijections

$$\frac{\dfrac{\text{cocones } \Gamma\widetilde{\Phi} \to B}{\text{cocones } \widetilde{\Gamma\Phi} \to B}}{\text{cocones } \Gamma\Phi \to B.}$$

Therefore $\varinjlim \Gamma\Phi \simeq \varinjlim \Gamma\widetilde{\Phi}$ for every $\mathcal{C}$-functor Γ, and by Theorem 5.2 this completes the proof

Let $\Psi : J \to A$ be another diagram and let $\mathcal{E}$ be a set of diagrams in J such that for each $E \in \mathcal{E}$, ΨE has a colimit in A and the cocone thus obtained is in $\mathcal{C}$. If $\widetilde{J}$ denotes the $\mathcal{E}$-cocompletion of J in the sense just described and $\widetilde{\Psi}$ the extension of Ψ to $\widetilde{J}$, then we have the following consequence of Theorem 6.1.

Corollary 6.2. *If* $\pi_0(-, \tilde{\Phi}) \simeq \pi_0(-, \tilde{\Psi})$, *then* $R_C\pi_0(-, \Phi) \simeq R_C\pi_0(-, \Psi)$.

Corollary 6.3. *If* $\pi_0(-, \tilde{\Phi})$ *is representable, then* Φ *has a* C*-absolute colimit.*

Corollary 6.4. *If* $\pi_0(-, \tilde{\Phi}) \simeq 1$, *then* Φ *is* C*-final.*

Proposition 6.5. *Let* I *have all finite coproducts and let* $\Phi : I \to A$ *preserve them. Then* $\pi_0(-, \Phi) : A \to S^{op}$ *preserves all finite coproducts which exist in* A.

Proof. Let $\amalg A_i$ be a finite coproduct in A. The canonical map $\pi_0(\amalg A_i, \Phi) \to \Pi\pi_0(A_i, \Phi)$ sends $[\amalg A_i \xrightarrow{\langle a_i \rangle} \Phi I]$ to $\langle [A_i \xrightarrow{a_i} \Phi I] \rangle$. It is easily checked that the map $\Pi\pi_0(A_i, \Phi) \to \pi_0(\amalg A_i, \Phi)$ which sends $\langle [A_i \xrightarrow{a_i} \Phi I_i] \rangle$ to $[\amalg A_i \xrightarrow{\amalg a_i} \amalg \Phi I_i \simeq \Phi(\amalg I_i)]$ is the inverse of the canonical map, as long as it is well defined, and this uses the finiteness of the coproduct.

For the remainder of the paper, we will assume that A has finite coproducts, and FCp will denote the set of cocones in A obtained from these coproducts.

Corollary 6.6. *If* $\tilde{\Phi}$ *is the extension of* $\Phi : I \to A$ *to the finite coproduct completion of* I, *then* $R_{FCp}\, \pi_0(-, \Phi) \simeq \pi_0(-, \tilde{\Phi})$.

Proof. By Theorem 6.1, $R_{FCp}\, \pi_0(-, \Phi) \simeq R_{FCp}\, \pi_0(-, \tilde{\Phi})$. Since a finite coproduct of finite coproducts is again a finite coproduct, the completion $\tilde{I}$ has all finite coproducts and $\tilde{\Phi}$ preserves them. Thus it follows from Proposition 6.5 that $R_{FCp}\, \pi_0(-, \tilde{\Phi}) \simeq \pi_0(-, \tilde{\Phi})$, and the result follows.

Corollary 6.7. Φ *has an* FCp*-absolute colimit if and only if* $\pi_0(-, \tilde{\Phi})$ *is representable, and* Φ *is* FCp*-final if and only if* $\pi_0(-, \tilde{\Phi}) = 1$.

Corollary 6.8. Φ *has a* FCp*-absolute colimit if and only if* $\tilde{\Phi}$ *has an absolute colimit, and* Φ *is* FCp*-final if and only if* $\tilde{\Phi}$ *is final.*

$\tilde{I}$ can be described more explicitly. An object of $\tilde{I}$ is a finite (and possibly empty) sequence of objects of I. A morphism $(I_1, ..., I_n) \to (I'_1, ..., I'_m)$ consists of a function $f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}$ and for each $i = 1, 2, ..., n$, a morphism $\alpha_i : I_i \to I'_{f(i)}$. The functor H sends an object I to the sequence of length one (I). Coproducts are formed by concatenation. $\tilde{\Phi}$ is defined by $\tilde{\Phi}(I_1, ..., I_n) = \amalg\Phi(I_i)$.

By using Corollary 6.8 and the characterization of [12] (given in the remark following Corollary 3.3), we obtain the following characterization for FCp-absolute colimits.

Theorem 6.9. *Let $\mu : \Phi \to A$ be a cocone. μ is a $\mathcal{F}\mathcal{C}p$-absolute colimit diagram if and only if there exists a finite number of objects of $\mathbf{I}$, $I_{0,1}, I_{0,2}, \ldots, I_{0,n}$ and a morphism $d_0 : A \to \coprod_i \Phi(I_{0,i})$ such that*

(a) $A \xrightarrow{d_0} \coprod_i \Phi(I_{0,i}) \xrightarrow{\langle \mu(I_{0,i})\rangle} A = A$;

(b) *for every $I \in |\mathbf{I}|$, there exist $I_{j,i} \in |\mathbf{I}|$, $j = 1, \ldots, k$ and $i = 1, 2, \ldots, n_j$ such that*

$$\begin{tikzcd}
\Phi I \arrow[r, "\mu I"] \arrow[d, equal] & A \arrow[r, "d_0"] & \coprod_i \Phi(I_{0,i}) \\
\Phi I \arrow[rr] \arrow[d, equal] & & \coprod_i \Phi(I_{1,i}) \arrow[u] \arrow[d] \\
\vdots \arrow[d, equal] & & \vdots \\
\Phi I \arrow[rr] \arrow[d, equal] & & \coprod_i \Phi(I_{n,i}) \arrow[u] \arrow[d] \\
\Phi I \arrow[rr, equal] & & \Phi I
\end{tikzcd}$$

commutes, where the horizontal arrows represent morphisms of $\mathbf{A}$ and where the vertical arrows on the right represent morphisms of the form

$$\begin{tikzcd}
\Phi(I_{j,i}) \arrow[r, "\mathrm{inj}_i"] \arrow[d, "\Phi(\alpha)"'] & \coprod_i \Phi(I_{j,i}) \arrow[d] \\
\Phi(I_{j+1,f(i)}) \arrow[r, "\mathrm{inj}_{f(i)}"'] & \coprod_i \Phi(I_{j+1,i})
\end{tikzcd}$$

for some function f of the indices.

Example 6.10. $A \underset{\mathrm{inj}_1}{\overset{\mathrm{inj}_2 \cdot a}{\rightrightarrows}} A + B \xrightarrow{\langle a, B\rangle} B$ is an $\mathcal{F}\mathcal{C}p$-absolute coequalizer which is not preserved by *all* functors.

Example 6.11. If $h : B \to D$, $f : A \to C + B$, $g : A \to C + D$ and $g = (C + h) \cdot f$ then

$$\begin{tikzcd}
A + B \arrow[r, "\langle f, \mathrm{inj}_2\rangle"] \arrow[d, "A+h"'] & C + B \arrow[d, "C+h"] \\
A + D \arrow[r, "\langle g, \mathrm{inj}_2\rangle"'] & C + D
\end{tikzcd}$$

is an $\mathcal{F}\mathcal{C}p$-absolute pushout which is not preserved by *all* functors. This example is due to Volger [15].

By using Corollary 6.7 we get the following characterization of $\mathsf{F}Cp$-final functors:

Theorem 6.12. *$\Phi : I \to A$ is $\mathsf{F}Cp$-final if and only if for every $A \in |A|$ there exist $I_1, \ldots, I_n \in |I|$ and a morphism*

$$a : A \to \coprod_i \Phi I_i$$

and for any $I'_1, I'_2, \ldots, I'_m \in |I|$ and $a' : A \to \coprod_i \Phi I'_i$ there is a finite chain

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, equal] & \coprod_i \Phi I_i \\
A \arrow[r] \arrow[d, equal] & \coprod_i \Phi I_i^{(1)} \arrow[u] \arrow[d] \\
\vdots \arrow[d, equal] & \vdots \\
A \arrow[r] \arrow[d, equal] & \coprod_i \Phi I_i^{(k)} \arrow[u] \arrow[d] \\
A \arrow[r, "a'"'] & \coprod_i \Phi I'_i
\end{tikzcd}$$

of the same nature as that of Theorem 6.9.

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