Solutions to Assignment 2

(1) Express each of the following numbers as a product of primes: 6, 24, 27, 35, 120.
Solution: $6 = 2{*}3$, $24 = 2^3{*}8$, $27 = 3^3$, $35 = 5{*}7$, $120 = 2^3{*}3{*}5$.

(2) Does a nonprime divided by a nonprime ever result in a prime? Does it ever result in a nonprime? Always? Sometimes? Never? Explain your answers.
Solution: A nonprime divided by a nonprime can result in a prime. For example, 12 (which is not prime) divided by 4 (which is not prime either) gives 3 (which is prime). A nonprime divided by a nonprime can also result in a nonprime. For example, 16 divided by 4 is 4.

(3) **For Extra Credit** Consider the following sequence of natural numbers: 1111, 11111, 111111, 1111111, 11111111, ... Are all these numbers prime? If not, can you describe infinitely many of these numbers that are definitely not prime?
Solution: They are not all prime. The numbers with an even number of digits can be factored as follows. $1111 = 11 * 101$, $111111 = 11 * 10101$, $11111111 = 11 * 1010101$, etc. The ones with a multiple of 3 as number of digits are divisible by 3.

(4) Suppose a certain number when divided by 13 yields a remainder of 7. If we add 22 to our original number, what is the remainder when this new number is divided by 13?
Solution: Let x be the original number. Then $x \equiv 7 \mod 13$. The new number is $x + 22$, and $x + 22 \equiv 7 + 22 \mod 13 \equiv 29 \mod 13 \equiv 3 \mod 13$.

(5) In this exercise and the next one we want to compare the number of perfect squares with the number of primes.
(a) How many perfect squares are less than or equal to 36?
Answer: 6 (namely, 1^2, 2^2, 3^2, 4^2, 5^2, and 6^2).
(b) How many are less than or equal to 144?
Answer: 12.
(c) In general, how many perfect squares are less than or equal to n^2?
Answer: n.
(d) Using these answers, estimate the number of perfect squares less than or equal to N (for any number N). [Hint: your estimate may involve square roots and should give the exact answer when N is itself a perfect square.]
Solution: It is $\lfloor \sqrt{N} \rfloor$ - that means: take the square root of N and then round down to get a whole number. If

N is a perfect square, say $N = n^2$, then this gives us $\lfloor\sqrt{n^2}\rfloor = \lfloor n\rfloor = n$, so we get the correct answer there. You can check this for yourself with a number that is not a perfect square. For example, lets take $N = 10$. $\lfloor\sqrt{10}\rfloor = \lfloor 3.16227766\ldots\rfloor = 3$. This is correct: there are three perfect squares below 10.

(6) Using a calculator or computer, fill in the last two columns of the following chart:

n	Number of primes up to n	Appr. number of squares up to n	(number of primes)/ (number of squares)
10	4	3	1.333333
100	25	10	2.5
1000	168	31	5.41935
10,000	1229	100	12.29
100,000	9592	316	30.3544
1,000,000	78,498	1,000	78.498
10,000,000	664,579	3162	210.1768
100,000,000	5,761,455	10,000	576.1455
1,000,000,000	50,847,534	31622	1607.9797

Given the information you have found, what do you conclude about the proportion of prime numbers to perfect squares? Are prime numbers more common than perfect squares or less common?

Answer: Prime numbers are more common, and as you increase the number N, prime numbers become increasingly more common.

For Extra Credit: Use the prime number theorem to conjecture a formula for the quotient of the number of primes up to n divided by the number of squares up to n.

Solution: The Prime Number Theorem states that the number of primes less than or equal to N is getting closer and closer to $N/\ln(N)$ as N increases. As we have seen, the number of perfect squares less than N is equal to $\lfloor\sqrt{N}\rfloor$ and this can be approximated by $\sqrt{N}$. So the quotient can be approximated by

$$\frac{N/\ln(N)}{\sqrt{N}} = \frac{N}{\ln(N)\sqrt{N}} = \frac{\sqrt{N}}{\ln(N)}.$$