Assignment 5 - Due Friday March 6

(1) **Discovering Fibonacci Relationships** By experimenting with numerous examples in search of a pattern, determine a simple formula for $(F_{n+1})^2 + (F_n)^2$ – that is, a formula for the sum of the squares of two consecutive Fibonacci numbers.

Solution The first Fibonacci numbers are: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, ... So we see that $F_2^2 + F_1^2 = 2 = F_3$, $F_3^2 + F_2^2 = 4 + 1 = 5 = F_5$, $F_4^2 + F_3^2 = 9 + 4 = 13 = F_7$, and we may guess that $(F_{n+1})^2 + F_n^2 = F_{2n+1}$.

(2) **A New Start** Suppose we build a sequence of numbers using the method of adding the previous two numbers to build the next one (just as for the Fibonacci sequence). This time, however, suppose our first two numbers are 2 and 1. Generate the first 15 terms. This sequence is called the *Lucas sequence* and is written as L_1, L_2, L_3, ... Compute the quotients of consecutive terms of the Lucas sequence as we did with the Fibonacci numbers. What number do these quotients approach?

What role do the initial values play in determining what number the quotients approach? In order to get an answer to this question, try two other first terms and generate a sequence. What do the quotients approach? Can you draw any conclusions?

Solution The first 15 Lucas numbers are: $L_1 = 2$, $L_2 = 1$, $L_3 = 3$, $L_4 = 4$, $L_5 = 7$, $L_6 = 11$, $L_7 = 18$, $L_8 = 29$, $L_9 = 47$, $L_{10} = 76$, $L_{11} = 123$, $L_{12} = 199$, $L_{13} = 322$, $L_{14} = 521$, $L_{15} = 843$.

The consecutive quotients are: $\frac{L_2}{L_1} = .5$, $\frac{L_3}{L_2} = 3$, $\frac{L_4}{L_3} \approx 1.3333$, $\frac{L_5}{L_4} = 1.75$, $\frac{L_6}{L_5} \approx 1.5714$, $\frac{L_7}{L_6} \approx 1.63636$, $\frac{L_8}{L_7} \approx 1.61111$, $\frac{L_9}{L_8} \approx 1.62069$, $\frac{L_{10}}{L_9} \approx 1.61702$, $\frac{L_{11}}{L_{10}} \approx 1.61842$

It looks as if these numbers approach Φ, the golden ratio.

Try this with different starting numbers: $Z_1 = 3$, $Z_2 = 6$ gives $Z_3 = 9$, $Z_4 = 15$, $Z_5 = 24$, $Z_6 = 39$, $Z_7 = 63$, $Z_8 = 102$, $Z_9 = 165$, $Z_{10} = 267$, $Z_{11} = 432$, ... Now the quotients are $\frac{Z_2}{Z_1} = 2$, $\frac{Z_3}{Z_2} = 1.5$, $\frac{Z_4}{Z_3} \approx 1.6667$, $\frac{Z_5}{Z_4} = 1.6$, $\frac{Z_6}{Z_5} = 1.625$, $\frac{Z_7}{Z_6} \approx$

1.6154, $\frac{Z_8}{Z_7} \approx 1.61905$, $\frac{Z_9}{Z_8} \approx 1.61765$, $\frac{Z_{10}}{Z_9} \approx 1.61818$, $\frac{Z_{11}}{Z_{10}} \approx 1.61798$ These fractions approximate the golden ratio again.

We may conjecture that this will always be the case, as long as your starting integers are not both zero.

Note: If you would like to give a mathematical proof (or argument) that shows that this is indeed the case, you can do that as follows: we are interested in the behaviour of $\frac{Z_{n+1}}{Z_n}$ as n grows large. That is the same as the behaviour of $\frac{Z_n}{Z_{n-1}}$ as n grows large. So lets assume that $\frac{Z_{n+1}}{Z_n}$ approaches some number ψ as n grows large (then $\frac{Z_n}{Z_{n-1}}$ appraoches ψ as well). Now $\frac{Z_{n+1}}{Z_n} = \frac{Z_n + Z_{n-1}}{Z_n} = \frac{Z_n}{Z_n} + \frac{Z_{n-1}}{Z_n} = 1 + \frac{Z_{n-1}}{Z_n}$. As n grows large, the left hand side approaches ψ, and the right hand side approaches $1 + \frac{1}{\psi}$, so $\psi = 1 + \frac{1}{\psi}$ and the only positive solution for this equation is the golden ratio.

Note that this argument only uses the equation $Z_{n+1} = Z_n + Z_{n-1}$, not the first two values. As long as they are not zero (so that we can actually divide by Fibonacci numbers), this argument will work.

(3) **Digging Up Fibonacci Roots** Using the square root key on a calculator, evaluate each number in the top row and record the answer in the bottom row of the following table:

Number	$\sqrt{\left(\frac{F_3}{F_1}\right)}$	$\sqrt{\left(\frac{F_4}{F_2}\right)}$	$\sqrt{\left(\frac{F_5}{F_3}\right)}$	$\sqrt{\left(\frac{F_6}{F_4}\right)}$	$\sqrt{\left(\frac{F_7}{F_5}\right)}$	$\sqrt{\left(\frac{F_8}{F_6}\right)}$	$\sqrt{\left(\frac{F_9}{F_7}\right)}$
Computed Value	1.4142	1.73205	1.5811	1.63299	1.61245	1.620185	1.617215

Looking at the chart, make a guess as to what special number $\sqrt{F_{n+2}/F_n}$ approaches as n gets larger and larger.

*Solution* We expect this sequence to approach the golden ratio.

Extra Credit Explain the result you found in the previous part.

Solution: $\frac{F_{n+2}}{F_n} = \frac{F_{n+2}F_{n+1}}{F_n F_{n+1}} = \frac{F_{n+2}}{F_{n+1}}\frac{F_{n+1}}{F_n}$ Both of the fractions on the right hand side approach the golden ratio Φ, so the product approaches Φ^2. So $\sqrt{\frac{F_{n+2}}{F_n}}$ approaches $\sqrt{\Phi^2} = \Phi$.
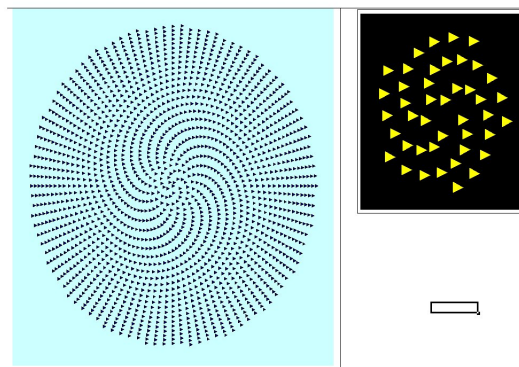
(4) **Flower Heads** Suppose that a flower head makes a .2 clockwise turn before releasing its next seed. Draw the shape of the seed head after 20 seeds have been released. Do the same for a

flowerhead that makes a .21 clockwise turn, and for a flowerhead
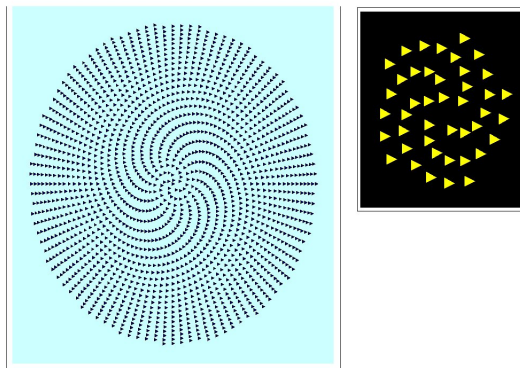that makes a .19 clockwise turn.

The 0.2 turn:
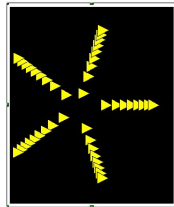
The 0.19 turn:

The 0.21 turn:

(5) **Golden Rectangles**

   (a) Explain what makes a rectangle a *Golden Rectangle.*
Solution Let the lengths of the sides of the rectangle be a and b with $a \geq b > 0$. Then the rectangle is golden if and only if $\frac{a}{b} = \Phi$.

   (b) Suppose that you have a golden rectangle and then you attach a square along its longer side to create a new rectangle. Is this new rectangle again golden? What if we repreat the process with the new larger rectangle? (For an image, see Page 245 in the book.)

     Solution: Suppose that the sides of the first rectangle are a and b and $a/b = \Phi$. Now you are adding a square of size a by a to the rectangle. The new rectangle has dimensions a (for the short side) and $a + b$ for the long side. The proportion of the new side lengths is $\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\Phi} = \Phi$, so the new rectangle is again golden.

(6) **Counterfeit Gold?** Draw a rectangle with its longer edge as the base (it could be a square, it could be a long and skinny rectangle, whatever you like, but take something that is not close to a Golden Rectangle). Now, using the top edge of large rectangle, draw the square just above the rectangle so that the square's base is the top edge of the rectangle. You have now produced a large new rectangle (the origional rectangle together with the square sitting above it). Now attach a square to the

right of this rectangle so that the square's left side is the right edge of the large rectangle. You've constructed an even larger rectangle. (for some images to guide you, see Page 246 of the book.)

Repeat this procedure - add a square to the top of the new rectangle, then add a square to the right, etc. Start with a small rectangle and conintue the process until you have almost filled the page. Now measure the dimensions of your final rectangle. What is the ratio of the side lengths? How does it compare to the Golden Ratio? Experiment with various starting rectangles. What do you notice about the ratios?

Solution: Everybody was able to execute this problem, so instead of giving you solutions, let me discuss this problem in a bit more detail.

Suppose that you start out with a rectangle with sides X_1 and X_2 with $X_2 \leq X_1 > 0$. Then the next rectangle has side lengths X_2 and $X_1 + X_2$, lets call that last length X_3, so $X_3 = X_2 + X_1$. The next rectangle will have side lengths X_3 and $X_3 + X_2$, which we will call X_4. So in general, we want to call $X_{n+1} = X_n + X_{n-1}$. Then the n-th rectangle you create will have side lengths X_n and X_{n+1}. The successive ratios of these side lengths is given by the sequence $\frac{X_2}{X_1}, \frac{X_3}{X_2}, \frac{X_4}{X_3}, \ldots, \frac{X_n}{X_{n-1}}, \ldots$ and we know from Problem 2 that this sequence will always approach the golden ratio.