Solutions to the problems on Assignment 5:

8.10(a) Claim: For every integer $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}$$

Proof: We prove this by induction on n. The induction basis is for $n = 2$:

$$\frac{1}{2+1} + \frac{1}{2*2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \geq \frac{7}{12}$$

This is correct, so the induction basis has been established.

Now assume the induction hypothesis:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}$$

and we need to show that this statement is true for $n+1$, *i.e.*,

$$\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2} \geq \frac{7}{12}$$

To prove this we start by manipulating the left hand side:

$$\begin{aligned}
&\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+2} = \\
&= -\frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \\
&\geq -\frac{1}{n+1} + \frac{7}{12} + \frac{1}{2n+1} + \frac{1}{2n+2} \\
&= \frac{7}{12} - \frac{2}{2n+2} + \frac{1}{2n+1} + \frac{1}{2n+2} \\
&= \frac{7}{12} + \frac{1}{2n+1} - \frac{1}{2n+2} \\
&= \frac{7}{12} + \frac{2n+2-2n-1}{(2n+1)(2n+2)} \\
&= \frac{7}{12} + \frac{1}{(2n+1)(2n+2)} \\
&\geq \frac{7}{12}
\end{aligned}$$

where the first inequality is an application of the induction hypothesis and the last one follows from the fact that $\frac{1}{(2n+1)(2n+2)} \geq 0$.

We conclude by induction that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{7}{12}$$

for all integers $n \geq 2$.

8.12 The proof of the induction step is flawed. If a and b are positive integers such that $\max(a, b) = n+1$, then a could be 1, while $b = n+1$, and in this case still $\max(a-1, b-1) = n$, $a-1$ is not a positive integer anymore, so the induction hypothesis cannot be applied.

8.13 Claim: n lines passing through a single point divide the plane into $2n$ regions.

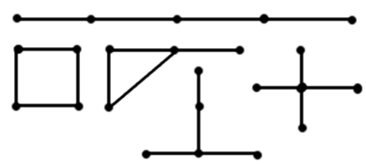
Proof: We do this by induction on the number of lines. For $n = 1$, we see that a single line divides the plane into $2 = 2 * 1$ regions, so the induction basis is correct.

The induction hypothesis says now that any arrangement of n lines passing through a single point divides the plane into $2n$ regions.

For the induction step we need to show that any $n + 1$ lines passing through a single point divide the plane into $2(n + 1)$ regions. So take any arrangement of $n+1$ lines passing through a single point. Now remove one of the lines temporarily. The remaining n lines pass through a single point, so they divide the plane into $2n$ regions by the induction hypothesis. Now we add the last line back in. It will cross through two regions which it will both divide into two new regions. So this causes the number of regions to be increased by 2. So the new number of regions is $2n + 2 = 2(n + 1)$.

We conclude by induction that n lines passing through a single point divide the plane into $2n$ regions.

9.5 The connected planar graphs with 4 edges are:



The connected planar graphs with 4 vertices are: