Polygonal Curvature

The Basic Definitions. A *polygonal surface* is a surface that is made out of convex (Euclidean) polygonal regions, which match in the sense that the intersection of two regions is either an edge or a vertex or empty. In this course you have seen various examples of polygonal surfaces already: the surfaces of the polyhedra we have studied, and the plane (when it has been tiled with polygonal tiles that match in the above described manner).

Depending on the sum of the angles of the regions meeting at a vertex, the surface may lie flat or it may curve in some way. We will now define the polygonal curvature at every point of a polygonal surface.

Definition 1. Let S be a polygonal surface. The *polygonal curvature* of S is 0 at every point of S that is in the interior of a polygonal region, or on the interior of an edge. A finite number of polygonal regions meet at every vertex v of S. Each of these regions has an interior angle with v as vertex. Let $s(v)$ be the sum of the radian measures of these interior angles. Then the *polygonal curvature of S at v* is $k(v) = 2\pi - s(v)$.

Example 2. If we express the plane as a polygonal surface through the regular tiling by equilateral triangles, there are six triangles meeting at each vertex v, so the angle sum is $s(v) = 6 * \frac{\pi}{3} = 2\pi$ and the curvature at v is $k(v) = 2\pi - 2\pi = 0$. This is what we would have expected since the plane is flat.

You may be a bit surprised by the fact that we set the curvature equal to 0 along an edge, even though two regions may meet at a non-zero angle along that edge. The reason is that if you would restrict yourself to a very small region around a point on an edge, you could straighten things out: the two polygonal regions meeting there could meet in a flat way. This is not the case at a vertex if the angle sum is not 2π.

Some Familiar Examples. Calculate the polygonal curvature at each vertex of the following surfaces:

(1) The icosahedron.
(2) The hexagonal tiling of the Euclidean plane.
(3) The soccer ball (the usual one, made up out of hexagons and pentagons).
(4) The dodecagedron.
(5) The octahedron.

A New Surface. Create a surface in the following way. Start with nine squares $S_{i,j}$, for $1 \leq i, j \leq 3$ and attach them to each other so that the right vertical edge of $S_{i,j}$ is the left vertical edge of $S_{i,j+1}$ (for $j = 1$ and $j = 2$ and all i), and the bottom horizontal edge of $S_{i,j}$ is the top horizontal edge of $S_{i+1,j}$ (for $i = 1$ and $i = 2$ and all j). Moreover, we also make the bottom horizontal edge of $S_{3,j}$ equal to the top horizontal edge of $S_{1,j}$.

(1) Make a sketch of this surface.
(2) What is the polygonal curvature of this surface at every internal vertex?

Now we want to further identify the left vertical edge of $S_{i,1}$ with the right vertical edge of $S_{i,3}$. Note that we cannot do this in $\mathbb{R}^3$ unless we are allowed to stretch our squares a bit, but it is possible to do this in $\mathbb{R}^4$ without stretching, so we will not worry about that. Now we have a closed surface again. Mathematicians call this a *torus*.

(1) Make a sketch of this surface.
(2) What is the polygonal curvature of this surface at every internal vertex?

Soccer Balls. If you start with a hexagonal tiling of the plane, you can turn it into a surface of positive curvature by replacing some hexagons by pentagons. If you do this in a symmetric way so that every pentagon is surrounded by hexagons and every hexagon has alternating pentagons and hexagons along its sides, you obtain the familiar soccer ball. Note that surfaces with positive curvature always want to close up on themselves.

If we want to create a surface with negative curvature we can start out with the same hexagonal tiling but instead of replacing tiles by regular pentagons, we will replace them by regular heptagons (polygons with seven edges). This way we create a surface that has been called a *hyperbolic soccer ball*.

(1) Make a supply of pentagons and hexagons with the templates provided and create a spherical soccer ball. (You may want to cut them with some extra flaps along the edges, so that you have something to glue along.)
(2) Make a supply of heptagons and hexagons with the templates provided and create a hyperbolic soccer ball. Make the surfaces at least 10 polygonal regions across. This will require a large supply of polygons, because the surface will get extremely wavy and floppy as it grows.

Lines and Triangles. Now we want to see what the effect of the curvature is on the geometry of the surface. For this part you will need a very long ruler, a piece of string and a small protractor to measure angles. (Ask for whatever you don't have and we will find something in the department.) First we will see what lines or curves with shortest distance look like on both surfaces.

(1) **Lines on the spherical soccer ball**
    (a) Take several pairs of two points on the spherical soccer ball and find the shortest path between them using your piece of string. Then draw those paths on the surface of the soccer ball.

   (b) If you continue those shortest paths in the same direction (how would you define that?) what do you get? You could think of these resulting paths as the "lines" of the geometry on the soccer ball.
   (c) Is it possible to make non-intersecting lines?
   (d) Is it possible to make lines with a common perpendicular (i.e., two lines for which there is a line which intersects both of them at a right angle)?
(2) **Lines on the hyperbolic soccer ball**
   To draw a line on the hyperbolic soccer ball choose two points that are far away from each other on the surface. Then pull the surface as straight as possible between those two points and use your long ruler to draw the line.
   (a) Construct several pairs of lines that admit a common perpendicular. Start by drawing a long line l on the hyperbolic soccer ball as instructed above. Then choose another point P outside the first line and halfway to the edge of your model. Construct a line m through P that is perpendicular to l. Then do the same thing again for m. Is it possible that the third line intersects l somewhere?
   (b) Experiment by doing this several times from different places. Is it possible to create two common perpendiculars between two lines? Write down some of your observations. How does this geometry differ from our usual Euclidean geometry?
   (c) Start with a line l and a perpendicular line m as before, with a point P on m. Is it possible to find lines through P that are not perpendicular to m, but that would definitely not intersect l, even if you extended the surface indefinitely? Draw some of those lines.
   (d) Measure the smallest angle with m at which you can find a line that doesn't intersect l. If you would extend your surface would you expect to find smaller angles?

Now we will explore what happens to triangles on both surfaces.

(1) **Triangles on the spherical soccer ball** Construct various triangles on the spherical soccer ball. What can you say about the sum of the angles in a triangle? How does it change with the size of the triangle? Make sure that you draw the triangles in such a way that there is enough of each angle in a polygonal so that you can measure the angles.
(2) **Triangles on the hyperbolic soccer ball** Construct various triangles on the hyperbolic soccer ball. What can you say about the sum of the angles in a triangle? How does it change with the size of the triangle? Make sure that you draw the triangles in such a way that there is enough of each angle in a polygonal so that you can

measure the angles.

Parallel Transport. Another way in which you can measure the effects of curvature on a surface is by a construction called 'parallel transport'. Here is a description of the process. Draw a triangle $\triangle ABC$ and pick a line l through vertex A. Construct the line l_1 through B such that l and l_1 make congruent corresponding angles with the line through A and B. Then construct a line l_2 through C such that l_2 and l_1 make congruent corresponding angles with the line through B and C. Finally, construct a line l_3 through A such that l_2 and l_3 make congruent corresponding angles with the line through C and A. Now we are interested in the angle between l and l_3.

(1) Take a flat piece of paper, and draw a Euclidean triangle $\triangle ABC$ on it. Draw a line l through A and do parallel transport. What is the resulting angle between l and l_3?
(2) Do the same experiment on a spherical soccer ball. You need to take your triangle big enough so that it doesn't just lie within one polygonal region. You may need a bigger ball than the one you had constructed. How do l and l_3 compare in this case?
(3) Do the same experiment on the hyperbolic soccer ball. Again you need to use a relatively large triangle for this experiment. How do l and l_3 compare in this case?
(4) Compare the results from the previous two parts. Did the line turn in the same direction for both experiments?

Area and Curvature. Yet another way to detect curvature on a surface is by considering the area of triangles and circular disks. If you want to do this part, you need to use the polygons with the rectangular grid to construct your soccer balls.

(1) Construct three triangles with a common angle on the surface of the hyperbolic soccer ball. Then calculate their area by counting the little squares inside the triangle. For complete hexagons you may take the area to be 41.57 and for complete heptagons you may take 58.14. For the rest you need to count squares (of size 1 by 1) and make guesses for parts of squares. For each of your triangles calculate both the area and the angle sum. Do you see a relationship?
(2) Construct two circles with the same center, but different radii. You can do this using a large compass or by using a piece of string with a pencil. How does the area change when you double the radius? How does the area change when you double the circumference?