Euler's Number for the Surface of a Donut

Discover Your World. How do we know that the world is round? A torus is the surface of a donut (or, bagel). Is it possible that we live on a torus, as opposed to a sphere? We have pictures of the earth that show uncontestably that it is a sphere. However, people knew this well before we were able to fly far enough away to check it. What was their evidence?

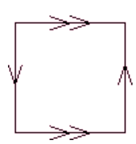
(1) What is the evidence you can think of for the fact that the surface of the earth is a sphere (this needs to be evidence that you can gather by remaining on that surface)?
(2) Suppose that you are on a planet and the surface is very lumpy and you are in a constant thick fog, how can you determine whether the surface of that planet is a sphere or possibly a torus? Would the kind of evidence you mentioned in the previous part work in that case?

If you are remaining at home you cannot answer these questions, you need to travel to find the answer. So the first suggestion is to start traveling with a large group of people - enough so that you can hold hands all the way around the diameter of the planet you are on. You want to make a large circle around the planet you find yourself on, and then you start all walking forward.

(1) What will happen if you are on a sphere?
(2) What could happen if you are on a torus? In which cases will you get a surprising result?

A space traveler lands on an unknown planet and wants to know its shape. He doesn't have enough people with him to perform the experiment mentioned before, so he just starts walking around and creates a large blue loop in the sand. Later he goes for a second walk and starts in a different place and makes a red loop in the sand. He is surprised by the end of his second walk: he only crossed the blue loop once! Is it possible that he was on a sphere? Is it possible that he was on the surface of a torus?

A Flat Model. Here is a way to create a torus out of flexible bendable material. Start with a square and glue the opposite sides together in such a way that the arrows match up:



If you first glue the top and the bottom you get a cylinder shaped surface. Notice that the other two segments have turned into circles and the arrows indicate how they need to be glued. You can do this by bending the cylinder until the two circles line up. This will then give you the torus surface.

This is very convenient, because it means that we can use the square as a kind of atlas map for the torus.

(1) To illustrate how this works, draw some loops on the surface of a torus and then draw the corresponding lines on the map which we will call the flat torus from now on).
(2) What are the rules for drawing continuing lines on the flat torus?
(3) If you want to get a feeling for how you need to work with the flat torus, try playing tic-tac-toe on this surface. There is a computer version of this game at www.math.ntnu.no/∼dundas/75060/TorusGames/html/TicTacToe.html Can you give a winning strategy for one of the players?
(4) Now draw two loops on the surface of the torus that intersect only once. Draw the corresponing curves on the flat torus. Verify that they are indeed only intersecting once.
(5) Note that the boundaries of the flat torus correspond to two circles on the surface of the torus. Now make a second map for the torus by cutting it open along two different circles. Draw the map images of the two loops from the previous question on this new flat torus. How different are they from the previous ones?

Graphs on the Surface of a Torus. In class we studied the question which graphs are planar. We learned that all planar graphs have Euler number equal to 1. For example, the complete graph on 5 vertices K_5 was shown to not be planar. We also saw that the utility graph connect three houses to gas, water, and power is not planar.

(1) Show that you can draw both of these graphs on the torus surface without intersecting edges.
(2) Draw the complete graphs K_7 and K_8 (on 7 and 8 vertices, respectively) on the torus without intersecting edges.
(3) Draw three graphs G_0, G_1, and G_3 with non-intersecting edges on the surface of the torus so that if you count $v - e + f$ you get 0, 1, and 2 respectively.

The answer to this last question may seem a bit disturbing. It seems to be saying that Euler's formula does not have the same value for all graphs on the torus. That is correct, but if we put an extra condition on our graphs, then the graphs that satisfy that condition will all have the same Euler number. We need to require that all the faces of the graph are *cells*. A region on a surface is a cell if all loops in that region are contractible within that region.

(1) Indicate in your examples above where you had a region that was not a cell.
(2) Give examples of regions on the torus that are not cells and of regions that are cells.

Finally, we want to show that for graphs on the torus for which all faces are cells, we have that $v - e + f = 0$. Before we can do that, we first need to quickly refresh our memory about spheres.

(1) Show that every graph without intersecting edges on a sphere satisfies the rule $v - e + f = 2$ (use what you have learned in class).
(2) Given a graph without intersecting edges on the torus, we want to change it into a graph on the sphere. Once you have drawn your graph, create a loop of edges from the graph that goes exactly once around the torus. Cut the surface of the torus around that loop of edges. and put a new face in each one of the two holes.
   (a) Show that the new surface can be deformed into a sphere and the new graph you have just created is a graph with non-intersecting edges on the sphere.
   (b) How is the Euler number of the new graph related to the Euler number of the old graph?
   (c) Use this to show that $v - e + f = 0$ for the original graph on the torus.

Finally we want to apply our results to show that K_8 cannot be drawn on the torus without intersecting edges.

(1) Show that $3f \leq 2e$ for K_8.
(2) Use $3f \leq 2e$ and $v - e + f = 0$ to show that K_8 cannot be drawn on the torus.
(3) Could you construct a surface where you can draw K_8 without intersecting edges?