Algebraic Infinitesimals

The algebraic infinitesimals $\mathbb{E}$ form an extension of the real numbers like the complex numbers. The number we add is called ε. It has the property that $\varepsilon^2 = 0$, but $\varepsilon \neq 0$. So when you work with algebraic infinitesimals, the equation $x^2 = 0$ has two solutions, namely, $x = 0$ and $x = \varepsilon$. We will call ε a pure infinitesimal. More general algebraic infinitesimals include the real numbers and they are closed under addition and multiplication. They are completely characterized by the fact that they are the smallest possible set with this property.

Addition and Multiplication. Your first task is to show that all algebraic infinitesimals are of the form $a + b\varepsilon$, where a and b are real numbers. Do this as follows:

(1) Show that every number of the form $a+b\varepsilon$ has to be an algebraic infinitesimal.
(2) Show that the set of numbers of the form $a + b\varepsilon$ if closed under addition and multiplication, by giving a formula for the addition and the multiplication of two algebraic infinitesimals $a+b\varepsilon$ and $c + d\varepsilon$.

The fact that the algebraic infinitesimals form the smallest set with these properties allows you to conclude that $\mathbb{E} = \{a + b\varepsilon | a, b \in \mathbb{R}\}$.

Division. Contrary to the case with the complex numbers, division of infinitesimals by infinitesimals can be a problem.

(1) Show that ε cannot have a multiplicative inverse, *i.e.*, show that there cannot be a number x such that $x\varepsilon = \varepsilon x$.
(2) Determine which algebraic infinitesimals do have a multiplicative inverse and give a formula for the inverse of such numbers $a + b\varepsilon$. (Hint: use conjugation.) Those are the numbers you can divide by.
(3) Calculate $\frac{a+b\varepsilon}{c+d\varepsilon}$ for an algebraic infinitesimal $c + d\varepsilon$ that has a multiplicative inverse.

Differentiation. Algebraic infinitesimals were first created in order to make differentiation better understandable. This works well for polynomials. Let $P(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ be a polynomial with $p_0, p_1, \ldots, p_n \in \mathbb{R}$.

(1) Show that $P(a + b\varepsilon) = P(a) + bP'(a)\varepsilon$, where P' is the derivative of P.

Transcendental Functions. From what we have learned above we can evaluate $f(a+b\varepsilon)$ for any rational function (as long as $a+b\varepsilon$ is chosen in such a way that the division is well-defined). The previous result about derivatives leads us to a way to define $f(a+b\varepsilon)$ for any function f for which we know the derivative:

$$f(a+b\varepsilon) := f(a) + bf'(a)\varepsilon.$$

(1) Verify that this gives the correct answer for the function $f(x) = \frac{1}{x}$.
(2) What is $e^{a+b\varepsilon}$?
(3) Use algebraic infinitesimals to prove the chain rule by calculating $f(g(a+b\varepsilon))$.
(4) Can you prove the product and the quotient rule in this way as well?
(5) Calculate $(a+b\varepsilon)^{c+d\varepsilon}$ (and determine when this is defined).

A Geometric Interpretation. Just as with the complex numbers we like to represent algebraic infinitesimals by points in the plane $\mathbb{R}^2$: the algebraic infinitesimal $a+b\varepsilon$ would correspond to the point (a,b) in the plane. We would like to have a kind of polar representation for these numbers that would work well for multiplication and addition. Note that $a+b\varepsilon = a(1+\frac{b}{a}\varepsilon)$. We will call a the *modulus* and $\frac{b}{a}$ (the slope of the vector (a,b)) the *angular part*.

(1) Describe the function e^x for algebraic infinitesimals in terms of moduli and angular parts.
(2) Give a geometric description of the algebraic infinitesimals with modulus 1. All complex numbers with modulus 1 can be written as $e^{i\theta}$, where θ is a real number. Is there a similar result for algebraic infinitesimals?
(3) Describe the addition, multiplication and division of two algebraic infinitesimals in terms of their moduli and angular parts. Can you give a geometric interpretation for these operations?