**Solutions to the problems from Assignment 3:**

**2** $(\sqrt{3}-i)^{10} = (2e^{-\frac{\pi i}{6}})^{10} = 2^{10}e^{-\frac{10\pi i}{6}} = 2^{10}e^{-\frac{5\pi i}{3}}$ so the *real part* is

$$2^{10}\cos(-\frac{5\pi}{3}) = 210 * \frac{1}{2} = 2^9 = 512,$$

and the *imaginary part* is

$$2^{10}\sin(-\frac{5\pi}{3}) = 2^{10} * \frac{\sqrt{3}}{2} = 2^9\sqrt{3} = 512\sqrt{3}.$$

$(\sqrt{3}-i)^{-7} = (2e^{-\frac{\pi i}{6}})^{-7} = 2^{-7}e^{\frac{7\pi i}{6}}$, so the *real part* is

$$2^{-7}\cos(\frac{7\pi}{6}) = -2^{-7}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2^8} = \frac{\sqrt{3}}{256},$$

and the *imaginary part* is

$$2^{-7}\sin(\frac{7\pi}{6}) = -2^{-7} * \frac{1}{2} = -\frac{1}{2^8} = -\frac{1}{256}.$$

$(\sqrt{3}-i)^n = (2e^{-\frac{\pi i}{6}})^n = 2^n e^{-\frac{n\pi i}{6}}$, so the imaginary part is $2^n\sin(-\frac{n\pi}{6})$. $(\sqrt{3}-i)^n$ is real if and only if its imaginary part is equal to zero. So $(\sqrt{3}-i)^n$ is real if and only if $2^n\sin(-\frac{n\pi}{6}) = 0$. This is zero if and only if $\sin(-\frac{n\pi}{6}) = 0$ and this is the case if and only if $\frac{n\pi}{6}$ is a (positive or neagtive) multiple of $\pi$, i.e., when $n$ is an integer multiple of 6.

**3 (a)** We first find one root of $z^{10} = i$. We do this by finding a value of $r$ and of $\theta$ such that $r^{10}e^{i\theta} = i = e^{\frac{\pi i}{2}}$. A solution is $r = 1$ and $\theta = \frac{\pi}{20}$. The other roots are found by multiplying by the 10th roots of unity. So the 10 roots of $z^{10} = i$ are:

$$\begin{aligned}
&e^{\frac{\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{2\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{2\pi i}{10}} = e^{\frac{5\pi i}{20}} = e^{\frac{\pi i}{4}} \\
e^{\frac{\pi i}{20}} e^{\frac{4\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{4\pi i}{10}} = e^{\frac{9\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{6\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{6\pi i}{10}} = e^{\frac{13\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{8\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{8\pi i}{10}} = e^{\frac{17\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{10\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{10\pi i}{10}} = e^{\frac{21\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{12\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{12\pi i}{10}} = e^{\frac{25\pi i}{20}} = e^{\frac{5\pi i}{4}} \\
e^{\frac{\pi i}{20}} e^{\frac{14\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{14\pi i}{10}} = e^{\frac{29\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{16\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{16\pi i}{10}} = e^{\frac{33\pi i}{20}} \\
e^{\frac{\pi i}{20}} e^{\frac{18\pi i}{10}} &= e^{\frac{\pi i}{20}+\frac{18\pi i}{10}} = e^{\frac{37\pi i}{20}}
\end{aligned}$$

These roots all lie on the unit circle, so the root closest to $i$ is the one with argument closest to $\frac{\pi}{2}$. That is $e^{\frac{9\pi i}{20}}$.

**3(b)** We need to find the 7 roots of $z^7 = \sqrt{3} - i$, i.e., $z^7 = 2e^{-\frac{\pi i}{6}}$. We start by finding one solution of $r^7 e^{i7\theta} = 2e^{-\frac{\pi i}{6}}$. A solution is $r = \sqrt[7]{2}$ and $\theta = -\frac{\pi}{6*7} = -\frac{\pi}{42}$, so one of the roots is $z = \sqrt[7]{2}e^{-\frac{\pi i}{42}}$. The other roots are found by multiplying by the 7th roots of unity. So the 7 roots of $z^7 = \sqrt{3} - i$ are:

$$\begin{aligned}
&\sqrt[7]{2}e^{-\frac{\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{2\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{12\pi i}{42}} = \sqrt[7]{2}e^{\frac{11\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{4\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{24\pi i}{42}} = \sqrt[7]{2}e^{\frac{23\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{6\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{36\pi i}{42}} = \sqrt[7]{2}e^{\frac{35\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{8\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{48\pi i}{42}} = \sqrt[7]{2}e^{\frac{47\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{10\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{60\pi i}{42}} = \sqrt[7]{2}e^{\frac{59\pi i}{42}} \\
\sqrt[7]{2}e^{-\frac{\pi i}{42}} * e^{\frac{12\pi i}{7}} &= \sqrt[7]{2}e^{-\frac{\pi i}{42}+\frac{72\pi i}{42}} = \sqrt[7]{2}e^{\frac{71\pi i}{42}}
\end{aligned}$$

The root closest to the imaginary axis has an argument closest to $\frac{\pi}{2} = \frac{21\pi}{42}$ or $\frac{3\pi}{2} = \frac{63\pi}{42}$. So the root closest to the imaginary axis is $\sqrt[7]{2}e^{\frac{23\pi i}{42}}$

**7** If you want to use the method of the book (which will work for all $\cos(n\theta)$), this goes as follows. First we derive from De Moivre's formula that

$$\cos(4\theta) + i\sin(4\theta) = (\cos(\theta) + i\sin(\theta))^4.$$

The right hand side of this equation can be expanded to

$$\begin{aligned}
(\cos(\theta) + i\sin(\theta))^4 &= \binom{4}{0}\cos^4(\theta) + \binom{4}{1}\cos^3(\theta)i\sin(\theta) \\
&\quad + \binom{4}{2}\cos^2(\theta)(i\sin(\theta))^2 \\
&\quad + \binom{4}{3}\cos(\theta)(i\sin(\theta))^3 + \binom{4}{4}(i\sin(\theta))^4 \\
&= \cos^4(\theta) + 4\cos^3(\theta)\sin(\theta)i - 6\cos^2(\theta)\sin^2(\theta) \\
&\quad - 4\cos(\theta)\sin^3(\theta)i + \sin^4(\theta) \\
&= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) \\
&\quad + (4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta))i.
\end{aligned}$$

By setting the real parts equal we obtain

$$\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$$

Since $\sin^2(\theta) = 1 - \cos^2(\theta)$, this gives us

$$\begin{aligned}
\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta)(1 - \cos^2(\theta)) + (1 - \cos^2(\theta))^2 \\
&= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + 1 - 2\cos^2(\theta) + \cos^4(\theta) \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1
\end{aligned}$$

So the answer to the first question is:

$$\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$$

When we substitute $\theta = \frac{\pi}{12}$ in this equation we get that

$$\cos(\frac{\pi}{3}) = 8\cos^4(\frac{\pi}{12}) - 8\cos^2(\frac{\pi}{12}) + 1$$

Note that $\cos(\frac{\pi}{3}) = \frac{1}{2}$, so this becomes

$$\frac{1}{2} = 8\cos^4(\frac{\pi}{12}) - 8\cos^2(\frac{\pi}{12}) + 1$$

Multiplying by 2 gives

$$1 = 16\cos^4(\frac{\pi}{12}) - 16\cos^2(\frac{\pi}{12}) + 2$$

and this is equivalent to

$$16\cos^4(\frac{\pi}{12}) - 16\cos^2(\frac{\pi}{12}) + 1 = 0$$

So $\cos(\frac{\pi}{12})$ is a root of the equation

$$16x^4 - 16x^2 + 1 = 0.$$

The other roots of this equation can be found by looking for angles $\theta$ such that $\cos(4\theta) = \frac{1}{2}$. (This is the only distinguishing feature of $\cos(\frac{\pi}{12})$ that makes this work.) So we look for $\theta$ such that

$$4\theta = \frac{\pi}{3} + 2k\pi.$$

This gives us $\theta = \frac{\pi}{12}$ with corresponding root $\cos(\frac{\pi}{12})$ (as we knew already) and $\theta = \frac{\pi}{12} + \frac{2\pi}{4} = \frac{7\pi}{12}$ with corresponding root $\cos(\frac{7\pi}{12})(= -\cos(\frac{5\pi}{12}))$, and $\theta = \frac{\pi}{12} + \frac{4\pi}{4} = \frac{13\pi}{12}$ with corresponding root $\cos(\frac{13\pi}{12}) = -\cos(\frac{\pi}{12})$, and $\theta = \frac{\pi}{12} + \frac{6\pi}{4} = \frac{19\pi}{12}$ with corresponding root $\cos(\frac{19\pi}{12}) = \cos(\frac{-5\pi}{12}) = \cos(\frac{5\pi}{12})$. So the roots of the equation are $\pm\cos(\frac{\pi}{12})$ and $\pm\cos(\frac{5\pi}{12})$. That gives us four roots. Since this is a quartic equation, we have found all of them.

**8** Since $|z| = 1$, we have that $z = \cos\theta + i\sin\theta$ for some angle $\theta$. Since $|z + \sqrt{2}| = 1$, we also have that $(\cos\theta + \sqrt{2})^2 + \sin^2\theta = 1$. This can be rewritten as $\cos^2\theta + 2\sqrt{2}\cos\theta + 2 + \sin^2\theta - 1 = 0$, and then as $2\sqrt{2}\cos\theta + 2 = 0$. So we find that $\cos\theta = -\frac{1}{\sqrt{2}}$. We conclude that $\theta = \frac{3\pi}{4}$ or $\theta = \frac{5\pi}{4}$, so the solutions are $e^{\frac{3i\pi}{4}}$ and $e^{\frac{5i\pi}{4}}$. And $(e^{\frac{3i\pi}{4}})^8 = e^{\frac{8*3i\pi}{4}} = e^{6i\pi} = 1$ and $(e^{\frac{5i\pi}{4}})^8 = e^{10i\pi} = 1$, so both solutions satisfy $z^8 = 1$.

**10** If $w$ is an $n$th root of unity, $w$ lies on the unit circle, so its modulus $|w| = 1$. So $\sqrt{w \cdot \overline{w}} = 1$, so $w \cdot \overline{w} = 1$. Note that we can divide by $w$ since zero is not a root of unity, so we get $\overline{w} = \frac{1}{w}$. (There are several other proofs of this fact.)

Now,

$$\begin{aligned}
\overline{(1-w)}^n &= (1-\overline{w})^n \text{ (since } \overline{1} = 1) \\
&= (1-\frac{1}{w})^n \text{ (by the statement above)} \\
&= (1-\frac{1}{w})^n w^n \text{ (since } w \text{ is an } n\text{th root of unity)} \\
&= ((1-\frac{1}{w})w)^n \\
&= (w-1)^n,
\end{aligned}$$

as required.

Finally,

$$\begin{aligned}(1-w)^{2n} &= (1-w)^n(1-w)^n \\ &= (1-w)^n(w-1)^n(-1)^n \\ &= (1-w)^n\overline{(1-w)}^n(-1)^n \\ &= ((1-w)\overline{(1-w)})^n(-1)^n,\end{aligned}$$

and for any complex number $z$, we have that $z \cdot \overline{z} = |z|^2$ is a real number, so $(1-w)\overline{(1-w)}$ is a real number, so $(1-w)^{2n}$ is a real number.