Solutions to Assignment 6

9.1 (a) Since each vertex lies on 3 edges, the number of edge-vertex pairs is equal to $2E$ and to $3V$, so

$$2E = 3V.$$

The number of face-edge pairs is equal to $4m+5n$ since there are m squares and n pentagons, but it can also be counted by the edges: every edge lies in two faces, so it is equal to $2E$. We conclude that

$$4m + 5n = 2E.$$

Finally since all faces are either squares or pentagons and there are m squares and n pentagons, we get that $m + n = F$.

(b) From Euler's formula we know that $V - E + F = 2$. With the first equation from the previous part we can transform this to $\frac{2}{3}E - E + F = 2$, so $-\frac{1}{3}E + F = 2$. Substituting the results form the other two equations above gives that $-\frac{1}{3}\frac{1}{2}(4m + 5n) + m + n = 2$. Multiplying both sides by 6 gives $-(4m + 5n) + 6m + 6n = 12$. So $2m + n = 12$, as required.

(c) There are the cube (for $m = 6$ and $n = 0$), the pentagonal prism (for $n = 2$ and $m = 5$), and the dodecahedron (for $m = 0$ and $n = 12$).

If you are wondering whether there are any other ones, you may check the list of Archimedean and Johnson polyhedra (you can find it on Wikipedia or on my office door). A more direct proof requires 3-dimensional trigonometry.

9.2 Consider a convex polyhedron with V vertices, E edges, and F faces. First count the number of edge-face pairs (e, f) where e is an edge that lies on the face f. Since every edge lies in exactly two faces, there are $2E$ such pairs. One the other hand, every face contains at least three edges, so the number of these pairs is greater than or equal to $3F$. We conclude that

$$2E \geq 3F.$$

Then count the number of vertex-edge pairs (v, e) where the vertex v lies on the edge e. Since each edge has precisely two vertices, there are $2E$ such pairs. However, for a convex polyhedron, each vertex lies on at least 3 edges, so the number such pairs is greater than or equal to $3V$, so

$$2E \geq 3V.$$

Euler's formula gives us that $V - E + F = 2$, hence

$$E = V + F - 2 \tag{1}$$

(1) together with $2E \geq 3F$ gives us that $2(V+F-2) \geq 3F$, so $2V+2F-4 \geq 3F$, so $2V \geq F + 4$ as required. (1) together with $2E \geq 3V$, this gives us that $2(V + F - 2) \geq 3V$, so $2V + 2F - 4 \geq 3V$, so $2F \geq V + 4$ as required.

We can also rewrite Euler's formula as

$$F = E - V + 2 \tag{2}$$

Combining (2) with $3F \leq 2E$ gives $3(E-V+2) \leq 2E$. So $3E-3V+6 \leq 2E$, so $-3V \leq -E - 6$, so $3V \geq E + 6$ as required.

Finally, we can rewrite Euler's formula as

(3) $$V = E - F + 2$$

Combining (3) with $3V \leq 2E$ gives $3(E-F+2) \leq 2E$, so $3E-3F+6 \leq 2E$, so $-3F \leq -E-6$, so$3F \geq E+6$, as required.

A polyhedron for which all these inequalities become equalities is the tetrahderon with $V = 4$, $E = 6$ and $F = 4$.

9.4 Since at least three edges meet at a vertex the number of vertex-edge pairs is greater than or equal to $3V$; it is also equal to $2E$, so $2E \geq 3V$, *i.e.*, $\frac{2}{3}E \geq V$. By counting face-edge pairs, we find that $4 * 9 + 8m = 2E$, *i.e.*, $36 + 8m = 2E$, so $18 + 4m = E$. Combined with the inequality from the previous sentence this gives us that

(4) $$V \leq \frac{2}{3}(18 + 4m).$$

Also, $F = 9 + m$. Euler's formula gives us that $V - E + F = 2$, so $V - (18 + 4m) + (9 + m) = 2$, so $V - 9 - 3m = 2$, so $V = 11 + 3m$. Substituting this in (4) gives $11 + 3m \leq \frac{2}{3}(18 + 4m)$. Rewriting gives $33 + 9m \leq 36 + 8m$, so $m \leq 3$. This contradicts the specification that $m \geq 4$. So this is not possible.

9.7 Claim: every connected planar graph has a vertex that is joined to at most five other vertices.

Proof: Towards a contradiction, assume that there is a (nonempty) plane graph where every vertex is joined to at least six other vertices. Count the vertex-edge pairs: since every vertex lies on at least 6 edges, there are at least $6v$ of them, so we have that $2e \geq 6v$, so $e \geq 3v$. Note that this graph has at least seven vertices and six edges, so we can apply the formula from Problem 3 of this chapter: $e \leq 3v - 6$, so we can conclude that $3v \leq e \leq 3v - 6$. This is a contradiction, so such a plane graph does not exist. Conclusion: every connected plane graph has a vertex that is joined to at most five other vertices.