ASYMPTOTICS OF THE WEIGHTED DELANNOY NUMBERS

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Abstract. The weighted Delannoy numbers give a weighted count of lattice paths starting at the origin and using only minimal east, north and northeast steps. Full asymptotic expansions exist for various diagonals of the weighted Delannoy numbers. In the particular case of the central weighted Delannoy numbers, certain weights give rise to asymptotic coefficients that lie in a number field. In this paper we apply a generalization of a method of Stoll and Haible to obtain divisibility properties for the asymptotic coefficients in this case. We also provide a similar result for a special case of the diagonal with slope 2.

1. Introduction

Asymptotic expansions of sequences are normally considered to be purely analytic objects. However, in several instances, when a sequence admitting an asymptotic expansion is of number theoretic origin, the coefficients that appear in the expansion possess striking arithmetic properties. Unexpectedly, the asymptotic coefficients are often restricted to all lie in a particular number field, and, in fact, have denominators that are only divisible by primes lying in some finite set. As a particular example, the central Delannoy numbers have an asymptotic expansion whose coefficients all lie in $\mathbb{Q}(\sqrt{2})$ and have denominators equal to some power of the prime $\sqrt{2}\mathbb{Z}[\sqrt{2}]$.

It is pleasing that if we start with number theoretic objects such as combinatorial sums then we obtain number theoretic objects in the expansion. Apart from the unpublished manuscript [12] of Stoll and Haible, this phenomenon has not been investigated and begs an explanation. In a previous paper [9], certain number fields that contain all of the asymptotic coefficients have been found for combinatorial sums related to weighted Delannoy numbers, and the present paper aims to further the understanding of these coefficients by determining a finite set of primes containing all those that can appear in their denominators.

Fix $\alpha, \beta, \gamma \in \mathbb{C}$. We consider paths that start at the origin, remain in the first quadrant and use only the steps $(1, 0)$ with weight α, $(0, 1)$ with weight β and $(1, 1)$ with weight γ. The weight of a path is then the product of the weights of the individual steps that comprise the path. For $r, s \in \mathbb{N}_0$, let $u_{r,s}$ denote the total of all of the weights of paths that connect the origin to the point (r, s). The $u_{r,s}$ are known as the weighted Delannoy numbers and are given by the recurrence relation

$$u_{r+1,s+1} = \alpha u_{r,s+1} + \beta u_{r+1,s} + \gamma u_{r,s} \qquad (r, s \geq 0)$$

Date: June 15, 2011.

Research supported by the Natural Sciences and Engineering Research Council of Canada and Killam Trusts.

subject to the initial conditions

$$u_{r,0} = \alpha^r \ (r \geq 0), \qquad u_{0,s} = \beta^s \ (s \geq 0).$$

We have the closed form expression

$$u_{r,s} = \sum_{k=0}^{r} \binom{r}{k} \binom{s}{k} \alpha^{r-k} \beta^{s-k} (\alpha\beta + \gamma)^k$$

(see [5, p. 87]).

In [9] a multivariate method of Pemantle and Wilson (see [10, 14]) was combined with a transfer method developed in the book of Flajolet and Sedgewick [4, Part B] to obtain asymptotic expansions for the univariate sequences $u_{r,ar}$ as $r \to \infty$ for $a \in \mathbb{N}$ in the case $\alpha\beta^a = 1$ and $\gamma/\alpha\beta \in \mathbb{Z}$. Inspection of the arguments used in [9] shows that in the case $a = 1$, we only require the condition $\gamma/\alpha\beta \in \mathbb{R}$. Since allowing $\alpha\beta \neq 1$ simply multiplies a corresponding binomial sum considered in [9] by $\alpha^r\beta^r$, full asymptotic expansions can be obtained for the central weighted Delannoy numbers $u_{r,r}$ in this more general setting.

The purpose of the present paper is to obtain divisibility properties for the asymptotic coefficients one obtains for $u_{r,r}$ in the case when $\alpha\beta \neq 0$ and $\gamma/\alpha\beta \in \mathbb{R} \cap \overline{\mathbb{Q}}$. We also treat the case of $u_{r,2r}$ when $\gamma = -9\alpha\beta$. In both cases we proceed by applying a generalized version (Proposition 4 below) of a method of Stoll and Haible that appears in [12]. The sequences of interest are then the central weighted Delannoy numbers $u_{r,r}$ given by

$$u_{r,r} = \sum_{k=0}^{r} \binom{r}{k}^2 \alpha^{r-k} \beta^{r-k} (\alpha\beta + \gamma)^k,$$

as well as the special case of $u_{r,2r}$, obtained by setting $\gamma = -9\alpha\beta$, that is given by

$$\alpha^r \beta^{2r} \sum_{k=0}^{r} (-1)^k \binom{r}{k} \binom{2r}{k} 8^k.$$

For the central weighted Delannoy numbers the case $\alpha\beta = 0$ leads to $u_{r,r} = \gamma^r$ which is not of interest. This is why we restrict our attention to the case $\alpha\beta \neq 0$. Further, with $d := 1 + \gamma/\alpha\beta$, if $d = 0$ then $u_{r,r} = \alpha^r\beta^r$. This case also fails to be of interest and so we assume that $d \neq 0$. In order to state the results, we will need to make clear what is meant by prime divisors and the denominator of an element lying in a number field. So let K be a number field with ring of integers $\mathcal{O}_K$ and $\delta \in K^*$. We have

$$\delta\mathcal{O}_K = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\delta)}$$

where the product is over all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ (the primes of K) and the uniquely determined exponents $v_{\mathfrak{p}}(\delta)$ are integers, all but finitely many of which are equal to zero. The prime divisors of δ are the primes $\mathfrak{p}$ of K for which $v_{\mathfrak{p}}(\delta) > 0$ and by the denominator of δ, we mean the product of $\mathfrak{p}^{-v_{\mathfrak{p}}(\delta)}$ over all primes $\mathfrak{p}$ of K for which $v_{\mathfrak{p}}(\delta) < 0$.

We have the following result which is proved in Section 3.

Proposition 1. *With the above notation and as $r \to \infty$ there exists an asymptotic expansion*

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right)$$

if $d > 0$, and

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \alpha^r \beta^r \frac{(1-\sqrt{d})^{2r+1}}{2\overline{\sqrt[4]{d}}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{r^\ell}\right)$$

if $d < 0$, where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$ and $\sqrt[4]{\cdot}$ denotes the principal branch of the fourth root. Further, if d is algebraic then the μ_ℓ are such that the only primes of $\mathbb{Q}(\sqrt{d})$ that can divide their denominators are the prime divisors of 2 and the prime divisors of $\sqrt{d}$.

The special case $\alpha\beta = \gamma = 1$ gives rise to the central Delannoy numbers. In [2], the first few coefficients of the resulting expansion are computed: As $r \to \infty$ we have

$$u_{r,r} \sim \frac{(3+2\sqrt{2})^r}{2\sqrt{(3\sqrt{2}-4)\pi r}} \left(1 - \frac{23}{16(8+3\sqrt{2})}\frac{1}{r} + \frac{2401}{1024(113+72\sqrt{2})}\frac{1}{r^2} + \ldots\right).$$

If we rewrite this as

$$u_{r,r} \sim \frac{(3+2\sqrt{2})^r}{2\sqrt{(3\sqrt{2}-4)\pi r}} \left(1 - \frac{8-3\sqrt{2}}{32r} + \frac{113-72\sqrt{2}}{1024r^2} + \ldots\right) \qquad (r \to \infty),$$

we see that the first few coefficients are equal to an element of $\mathbb{Z}[\sqrt{2}]$ divided by a power of 2. If we factor these coefficients over $\mathbb{Q}(\sqrt{2})$, we see that $\sqrt{2}\mathbb{Z}[\sqrt{2}]$ is the only prime that divides the denominators of these coefficients. Proposition 1 tells us that this pattern continues so that, once factored over $\mathbb{Q}(\sqrt{2})$, the denominators of the asymptotic coefficients are all powers of $\sqrt{2}\mathbb{Z}[\sqrt{2}]$.

We can also obtain divisibility properties for the asymptotic coefficients in the case of $u_{r,2r}$ when $\gamma = -9\alpha\beta$. The result, which is also proved in Section 3, is the following.

Proposition 2. *With the above notation, set $\gamma = -9\alpha\beta$. There exist constants μ_ℓ, $\eta_\ell \in \mathbb{Q}$ for $\ell \in \mathbb{N}$, the denominators of which are divisible only by the primes 2 and 3 such that, as $r \to \infty$,*

$$u_{r,2r} \sim \frac{(-27\alpha\beta^2)^r}{2^{2/3}\Gamma(2/3)r^{1/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{(-27\alpha\beta^2)^r}{2^{4/3}\Gamma(1/3)r^{2/3}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\eta_\ell}{r^\ell}\right).$$

For a general discussion of asymptotics of lattice paths see [1]. For more on the Delannoy numbers, see [2, 3, 11, 13] and for more on weighted lattice paths see [5, 6].

2. A generalization of the method of Stoll and Haible

Fix $\varphi \in \mathbb{Q}$ and $q \in \mathbb{N}$. Let $\mathcal{F}$ denote the $\mathbb{C}$-vector space of all generating functions $F(x) = \sum_{r=0}^{\infty} f_r x^r \in \mathbb{C}[\![x]\!]$ such that f_r admits a full asymptotic expansion of the form

$$f_r \sim r^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{r^{m/q}} \qquad (r \to \infty) \tag{1}$$

for some integer N and sequence $\{a_m\}_{m \geq N} \subseteq \mathbb{C}$. With this notation, and denoting the space of all finite-tailed Laurent series by the usual $\mathbb{C}(\!(x)\!)$, we define a $\mathbb{C}$-linear transformation $\Psi : \mathcal{F} \to x^{\varphi}\mathbb{C}(\!(x^{1/q})\!)$ as follows. Given $F \in \mathcal{F}$ with coefficient sequence $\{f_r\}_{r \geq 0}$ satisfying (1), we set

$$\Psi(F) = \sum_{k=N}^{\infty} \frac{a_k}{\Gamma(\varphi + k/q)} \log(1+x)^{\varphi + k/q - 1} \in x^{\varphi}\mathbb{C}(\!(x^{1/q})\!).$$

Here we are considering division by Γ as being defined to be multiplication by the entire function $1/\Gamma$. Our transformation Ψ is therefore well-defined. The following result was proved in [12, Theorem 2] for $\varphi = 0$, but the proof of the general case is completely analogous. For completeness, we include the proof below.

Proposition 3. *With the above notation, the linear transformation Ψ satisfies the following properties.*

(a) $\Psi(xF(x)) = (x+1)\Psi(F(x))$.
(b) $\Psi\left(\frac{d}{dx}F(x)\right) = \frac{d}{dx}\Psi(F(x))$.
(c) *If F is a polynomial then* $\Psi(F(x)) = 0$.

Proof. Let $F(x)$ have coefficient sequence $\{f_r\}_{r \geq 0}$ that satisfies (1).

(a) Defining $f_{-1} := 0$, the generating function $xF(x)$ has coefficient sequence $\{f_{r-1}\}_r$ satisfying

$$f_{r-1} \sim (r-1)^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{(r-1)^{m/q}} \qquad (r \to \infty).$$

We now rewrite this asymptotic series in terms of r rather than $r-1$. We find that

$$\begin{aligned}
(r-1)^{-\varphi} \sum_{m=N}^{\infty} \frac{a_m}{(r-1)^{m/q}} &= \sum_{m=N}^{\infty} a_m r^{-\varphi - m/q} \left(1 - \frac{1}{r}\right)^{-\varphi - m/q} \\
&= \sum_{m=N}^{\infty} a_m r^{-\varphi - m/q} \sum_{j=0}^{\infty} \binom{-\varphi - m/q}{j} (-1)^j r^{-j} \\
&= r^{-\varphi} \sum_{m=N}^{\infty} \sum_{j=0}^{\lfloor \frac{m-N}{q} \rfloor} a_{m-qj} \binom{\varphi + m/q - 1}{j} r^{-m/q}.
\end{aligned}$$

Therefore

$$\Psi(xF(x)) = \sum_{k=N}^{\infty} \sum_{j=0}^{\lfloor \frac{k-N}{q} \rfloor} a_{k-qj} \binom{\varphi + k/q - 1}{j} \frac{1}{\Gamma(\varphi + k/q)} \log(1+x)^{\varphi + k/q - 1}.$$

On the other hand, we have

$$\begin{aligned}(x+1)\Psi(F(x)) &= \exp(\log(1+x))\Psi(F(x)) \\ &= \left(\sum_{k=0}^{\infty} \frac{\log(1+x)^k}{k!}\right)\left(\sum_{k=N}^{\infty} \frac{a_k}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1}\right) \\ &= \sum_{k=N}^{\infty} \sum_{j=0}^{\lfloor \frac{k-N}{q} \rfloor} \frac{a_{k-qj}}{j!\Gamma(\varphi+k/q-j)} \log(1+x)^{\varphi+k/q-1}.\end{aligned}$$

The proof is then completed by noticing that

$$\binom{\varphi+k/q-1}{j}\frac{1}{\Gamma(\varphi+k/q)} = \frac{1}{j!\Gamma(\varphi+k/q-j)}.$$

(b) From (a), it is sufficient to verify that $\Psi\left(x\frac{d}{dx}F(x)\right) = (x+1)\frac{d}{dx}\Psi(F(x))$. To this end, we start by noticing that the coefficient sequence of $x\frac{d}{dx}F(x)$ is $\{rf_r\}_r$, having asymptotic expansion

$$rf_r \sim r^{-\varphi} \sum_{m=N-q}^{\infty} \frac{a_{m+q}}{r^{m/q}} \qquad (r \to \infty).$$

Therefore

$$\Psi\left(x\frac{d}{dx}F(x)\right) = \sum_{k=N-q}^{\infty} \frac{a_{k+q}}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1}.$$

On the other hand, we have

$$\begin{aligned}(x+1)\frac{d}{dx}\Psi(F(x)) &= (x+1)\sum_{k=N}^{\infty} \frac{a_k(\varphi+k/q-1)}{\Gamma(\varphi+k/q)} \frac{\log(1+x)^{\varphi+k/q-2}}{1+x} \\ &= \sum_{k=N-q}^{\infty} \frac{a_{k+q}}{\Gamma(\varphi+k/q)} \log(1+x)^{\varphi+k/q-1},\end{aligned}$$

where for the last equality we used the basic identity $\Gamma(z+1) = z\Gamma(z)$.

(c) If F is a polynomial, then the sequence $\{f_r\}$ eventually consists of all zero terms and so each of the a_m is equal to zero. □

By induction we can conclude from Proposition 3 that if $F(x)$ is such that $L_x(F)$ is a polynomial for some linear differential operator L_x with polynomial coefficients, then $\Psi(F(x))$ satisfies the linear differential operator L_{x+1}.

Before stating the main result of this section we require a couple of lemmas. Firstly, we have the following result.

Lemma 1. *Let $r \in \mathbb{Q}$, K be a number field and $\mathfrak{p}$ be a prime of K. If $v_{\mathfrak{p}}(r) \geq 0$ then $v_{\mathfrak{p}}(\binom{r}{n}) \geq 0$ for all $n \in \mathbb{N}_0$.*

Proof. Let p be the prime lying below $\mathfrak{p}$. Since $v_{\mathfrak{p}}(r) \geq 0$, we have also $v_p(r) \geq 0$. Consequently, r is a p-adic integer. It follows that for any $n \in \mathbb{N}_0$, $\binom{r}{n}$ is also a p-adic integer (see, e.g., [7, Lemma 4.3.9]). Since $v_p(\binom{r}{n}) \geq 0$, we conclude that $v_{\mathfrak{p}}(\binom{r}{n}) \geq 0$ as well. □

The following lemma is based on [12, Lemma 4] and again follows by an entirely analogous proof. As before, we include the proof for completeness.

Lemma 2. *Let $\theta \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and K be a number field. Then*

$$(e^x - 1)^\theta = \sum_{n=0}^{\infty} \frac{s_n(\theta)}{(\theta+1)\dots(\theta+n)} x^{\theta+n} \tag{2}$$

with

$$s_n(x) = \sum_{0 \le k \le m \le n} \binom{x+n}{m+n} \binom{m+n}{k+n} (-1)^{m-k} S(n+k,k) \in \mathbb{Q}[x], \tag{3}$$

where $S(a,b)$ denotes the appropriate Stirling number of the second kind. Further, the only primes of K that can divide the denominators of the $s_n(\theta)$ are the primes dividing the denominator of θ.

Proof. We have

$$\frac{(e^x-1)^\theta}{x^\theta} = \left(1 + \left(\frac{e^x-1}{x} - 1\right)\right)^\theta = \sum_{m=0}^{\infty} \frac{\theta(\theta-1)\dots(\theta-m+1)}{m!} \left(\frac{e^x-1}{x} - 1\right)^m \tag{4}$$

which lies in $\mathbb{Q}[\![x]\!]$. We can therefore define $s_n(\theta)$ by (2). We now show that (3) holds by comparing the coefficient of $x^{\theta+n}$ in (2) with the coefficient of x^n in (4). On the one hand, from (2) we see that this coefficient is

$$\frac{s_n(\theta)}{(\theta+1)\dots(\theta+n)}. \tag{5}$$

On the other hand, looking at (4), noticing that only the terms having $m \le n$ contribute to the coefficient of x^n, and letting $[x^n]$ be the function that extracts the coefficient of x^n, the coefficient is given by

$$\begin{aligned} &\sum_{m=0}^{n} \frac{\theta(\theta-1)\dots(\theta-m+1)}{m!} [x^n] \left(\frac{e^x-1}{x} - 1\right)^m \\ &= \sum_{m=0}^{n} \frac{\theta(\theta-1)\dots(\theta-m+1)}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} [x^n] \left(\frac{e^x-1}{x}\right)^k \\ &= \sum_{0 \le k \le m \le n} \frac{\theta(\theta-1)\dots(\theta-m+1)}{m!} \binom{m}{k} (-1)^{m-k} S(n+k,k) \frac{k!}{(n+k)!}. \end{aligned} \tag{6}$$

Here, we have used [12, p. 6, (2)] that implies that for $k \in \mathbb{N}_0$,

$$\frac{(e^x-1)^k}{k!} = \sum_{\ell=0}^{\infty} S(\ell,k) \frac{x^\ell}{\ell!},$$

in order to obtain (6). Setting (5) equal to (6) yields (3). For the second part, suppose that $\mathfrak{p}$ is a prime of K such that $v_\mathfrak{p}(\theta) \ge 0$. Then, for all integers, n, $v_\mathfrak{p}(\theta+n) \ge 0$. We can therefore apply Lemma 1 to conclude that $v_p\binom{\theta+n}{m+n} \ge 0$ for all integers m and n. Since the rest of the components of (3) are $\mathfrak{p}$-adic integers, we conclude that $v_\mathfrak{p}(s_n(\theta)) \ge 0$. This completes the proof. □

In Lemma 2, it is also possible to put a lower bound on the valuation $v_\mathfrak{p}(s_n(\theta))$ of $s_n(\theta)$ at primes $\mathfrak{p}$ that divide the denominator of θ, but for simplicity this was not included.

Before stating the main result of this section, we observe that the definition of $\Psi(F(x))$ yields

$$\Psi(F(x)) = \frac{x^{\varphi+N/q-1}}{\Gamma(\varphi+N/q)} \sum_{n=0}^{\infty} b_n x^{n/q} \tag{7}$$

where, in particular,

$$b_\ell = \frac{\Gamma(\varphi+N/q)}{\Gamma(\varphi+(N+\ell)/q)} a_{\ell+N} \qquad (0 \leq \ell < q). \tag{8}$$

Here we make the assumption

$$\varphi + \frac{N+\ell}{q} \notin \mathbb{Z}_{\leq 0} \qquad (0 \leq \ell < q) \tag{9}$$

in order to have well-defined and nonzero quotients appearing in (8).

We have now arrived at the main result of this section. The special case where $K = \mathbb{Q}$ and $q = 1$ appears as [12, Corollary 5].

Proposition 4. *With the above notation, let K be a number field and $0 \leq \ell < q$. If $b_\ell \neq 0$ then $a_{N+\ell} \neq 0$. In this case, if $b_{qn+\ell}/b_\ell \in K$ for all n then $a_{qk+N+\ell}/a_{N+\ell} \in K$ for all k and the only primes of K that can divide the denominator of $a_{qk+N+\ell}/a_{N+\ell}$ are the primes that divide the denominator of $\varphi+N/q+\ell/q$ and the primes that divide the denominator of $n!b_{qn+\ell}/b_\ell$ for some $0 \leq n \leq k$.*

Again, it is possible to give lower bounds on the valuations of the asymptotic coefficients at the primes of K appearing in Proposition 4, but for simplicity this was not included.

Proof of Proposition 4. Define $B(x) = \Psi(F(x))$ and $A(x) = B(e^x - 1)$. Then

$$A(x) = \sum_{k=0}^{\infty} \frac{a_{k+N}}{\Gamma(\varphi+k/q+N/q)} x^{\varphi+k/q+N/q-1}. \tag{10}$$

By (7) we have

$$A(x) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(\varphi+N/q)} (e^x-1)^{n/q+\varphi+N/q-1}.$$

By (9) we can apply Lemma 2 to obtain the following expression for $A(x)$:

$$\sum_{n=0}^{\infty} \frac{b_n}{\Gamma(\varphi+N/q)} \sum_{m=0}^{\infty} \frac{s_m(n/q+\varphi+N/q-1) x^{n/q+\varphi+N/q-1+m}}{(n/q+\varphi+N/q)\dots(n/q+\varphi+N/q-1+m)}. \tag{11}$$

Comparing the coefficient of $x^{\varphi+k/q+N/q-1}$ in (10) with that of (11) we find that the quotient $\frac{a_{k+N}}{\Gamma(\varphi+k/q+N/q)}$ is given by

$$\sum_{n+mq=k} \frac{b_n}{\Gamma(\varphi+N/q)} \frac{s_m(n/q+\varphi+N/q-1)}{(n/q+\varphi+N/q)\dots(n/q+\varphi+N/q-1+m)}.$$

Thus, for $k \geq 0$,

$$\begin{aligned} a_{k+N} &= \sum_{n+mq=k} \frac{\Gamma(\varphi + N/q + n/q)}{\Gamma(\varphi + N/q)} b_n s_m(\varphi + N/q + n/q - 1) \\ &= \sum_{m=0}^{\lfloor k/q \rfloor} \frac{\Gamma(\varphi + N/q + k/q - m)}{\Gamma(\varphi + N/q)} b_{k-mq} s_m(\varphi + N/q + k/q - m - 1). \end{aligned}$$

Replacing k with $kq + \ell$ we see that the quotient $a_{kq+\ell+N}/a_{\ell+N}$ is given by

$$\begin{aligned} &\frac{\Gamma(\varphi + N/q)}{\Gamma(\varphi + N/q + \ell/q) b_\ell} \sum_{m=0}^{k} \frac{\Gamma(\varphi + N/q + k + \ell/q - m)}{\Gamma(\varphi + N/q)} \\ &\qquad\qquad \times b_{(k-m)q+\ell} s_m(\varphi + N/q + k + \ell/q - m - 1) \\ &= \sum_{m=0}^{k} \frac{\Gamma(\varphi + N/q + \ell/q + m)}{\Gamma(\varphi + N/q + \ell/q)} \frac{b_{mq+\ell}}{b_\ell} s_{k-m}(\varphi + N/q + \ell/q + m - 1) \\ &= \sum_{m=0}^{k} \binom{\varphi + N/q + \ell/q + m - 1}{m} m! \frac{b_{mq+\ell}}{b_\ell} s_{k-m}(\varphi + N/q + \ell/q + m - 1). \end{aligned}$$

The proof now follows from Lemmas 1 and 2. □

In the case of the central weighted Delannoy numbers, we can obtain enough information regarding the coefficients b_n to apply Proposition 4 to obtain meaningful divisibility properties for the original asymptotic coefficients a_m. Another case where we can do this is for $u_{r,2r}$ in the case $\gamma = -9\alpha\beta$. For purposes of clarity, we now state the special cases of Proposition 4 that apply to these two situations.

Corollary 1. *With the above notation, let K be a number field. Suppose further that $\{b_n\}_n$ is defined by*

$$\Psi(F(x)) = \frac{1}{\sqrt{\pi x}} \left(1 + \sum_{n=1}^{\infty} b_n x^n \right),$$

where each $b_n \in K$. Then the coefficients a_k for $k \geq 0$ all lie in K and the only primes that can divide their denominators are the primes dividing 2 and the primes dividing the denominator of some $n!b_n$.

Corollary 2. *With the above notation, let K be a number field. Suppose further that $\{c_n\}_n$ and $\{d_n\}_n$ are defined by*

$$\Psi(F(x)) = \frac{x^{-2/3}}{\Gamma(1/3)} \left(1 + \sum_{n=1}^{\infty} c_n x^n \right) + b x^{-1/3} \left(1 + \sum_{n=1}^{\infty} d_n x^n \right),$$

where each $c_n, d_n \in K$ and $b \in \mathbb{C}$ is nonzero. Then the coefficients $a_{3k}, a_{3k+1}/a_1$ for $k \geq 0$ all lie in K and the only primes that can divide their denominators are the primes dividing 3 or the denominator of some $n!c_n$ or the denominator of some $n!d_n$.

We note that in Corollary 2, we can say more. The only primes that can divide the denominators of the coefficients a_{3n} are the primes that divide 3 or the denominator of some $n!c_n$. Similarly, the only primes that can divide the denominators of the a_{3n+1}/a_1 are the primes $\mathfrak{p}$ such that $v_{\mathfrak{p}}(2) < v_{\mathfrak{p}}(3)$ and the primes that divide the denominator of some $n!d_n$.

3. The proofs of Propositions 1 and 2

In [9] a multivariate method of Pemantle and Wilson developed in [10] was combined with a transfer method developed in the book of Flajolet and Sedgewick [4] in order to obtain full asymptotic expansions for various diagonals of weighted Delannoy numbers. Combining Propositions 6 and 7 of [9], we obtain the following result.

Proposition 5. *Let $d \in \mathbb{Z}$ be nonzero. As $r \to \infty$ there exists an asymptotic expansion*

$$\sum_{k=0}^{r} \binom{r}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right)$$

if $d > 0$, and

$$\sum_{k=0}^{r} \binom{r}{k}^2 d^k \sim \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \frac{(1-\sqrt{d})^{2r+1}}{2\overline{\sqrt[4]{d}}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{r^\ell}\right)$$

if $d < 0$, where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$ and $\sqrt[4]{\cdot}$ denotes the principal branch of the fourth root.

Inspection of the proofs of Propositions 6 and 7 of [9] shows that in Proposition 5, one does not require $d \in \mathbb{Z}$ and so setting $d = 1 + \frac{\gamma}{\alpha\beta}$, and using the fact that

$$u_{r,r} = \alpha^r \beta^r \sum_{k=0}^{r} \binom{r}{k}^2 d^k$$

we obtain, as $r \to \infty$,

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right)$$

if $d > 0$, and

$$u_{r,r} \sim \alpha^r \beta^r \frac{(1+\sqrt{d})^{2r+1}}{2\sqrt[4]{d}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\mu_\ell}{r^\ell}\right) + \alpha^r \beta^r \frac{(1-\sqrt{d})^{2r+1}}{2\overline{\sqrt[4]{d}}\sqrt{\pi r}} \left(1 + \sum_{\ell=1}^{\infty} \frac{\overline{\mu_\ell}}{r^\ell}\right)$$

if $d < 0$, where the constants $\mu_\ell \in \mathbb{Q}(\sqrt{d})$. We are left with proving that if d is algebraic, then the μ_ℓ are such that the only primes of $\mathbb{Q}(\sqrt{d})$ that can divide their denominators are the prime divisors of 2 and the prime divisors of $\sqrt{d}$. We will invoke Corollary 1 to accomplish this.

Define G to be the generating function of $\{u_{r,r}/\alpha^r\beta^r\}_r$ and F to be the generating function of

$$f_r := \frac{2\sqrt[4]{d}\sqrt{\pi}u_{r,r}}{\alpha^r\beta^r(\sqrt{d}+1)^{2r+1}}.$$

A calculation using the computer algebra system Maple 11 [8] verifies that the generating function G satisfies the linear ordinary differential equation given by

$$((1-d)^2x^2 - 2(1+d)x + 1)G'(x) + ((1-d)^2x - (1+d))G(x) = 0. \tag{12}$$

Since the generating function for $\{f_r\}_r$ is given by

$$F(x) = \frac{2\sqrt[4]{d}\sqrt{\pi}}{\sqrt{d}+1} G\left(\frac{x}{(\sqrt{d}+1)^2}\right)$$

we can derive a differential equation satisfied by F from the differential equation (12) satisfied by G. After simplifying, we find that F satisfies the following ordinary differential equation:

$$((1-\sqrt{d})^2x^2 - 2(1+d)x + (1+\sqrt{d})^2)F'(x) + ((1-\sqrt{d})^2x - (1+d))F(x) = 0.$$

It follows that B defined by $B(x) = \Psi(F(x))$ satisfies the ordinary differential equation

$$((1-\sqrt{d})^2(x+1)^2 - 2(1+d)(x+1) + (1+\sqrt{d})^2)B'(x) \\ +((1-\sqrt{d})^2(x+1) - (1+d))B(x) = 0.$$

Solving this equation for B we obtain

$$B(x) = \frac{C}{\sqrt{x(4\sqrt{d} - (\sqrt{d}-1)^2x)}}$$

for some constant C. But from (7) and (8) together with the fact that $a_0 = 1$, we see that

$$B(x) = \frac{1}{\sqrt{\pi x}}\left(1 + \sum_{r=1}^{\infty} b_r x^r\right). \tag{13}$$

We conclude that

$$C = \frac{2\sqrt[4]{d}}{\sqrt{\pi}}.$$

Substituting in this value for C yields

$$B(x) = \frac{2\sqrt[4]{d}}{\sqrt{\pi x}\sqrt{4\sqrt{d} - (\sqrt{d}-1)^2 x}} = \frac{1}{\sqrt{\pi x}}\left(1 - \frac{(\sqrt{d}-1)^2}{4\sqrt{d}}x\right)^{-1/2}. \tag{14}$$

Comparing the right-hand sides of (13) and (14) yields

$$b_r = \binom{-1/2}{r}(-1)^r\delta^r = \binom{r-1/2}{r}\delta^r \qquad \text{where} \qquad \delta = \frac{(\sqrt{d}-1)^2}{4\sqrt{d}}.$$

We conclude from Corollary 1 that the only primes of $\mathbb{Q}(\sqrt{d})$ that can divide the denominators of the asymptotic coefficients are the prime divisors of 2 and $\sqrt{d}$. This completes the proof of Proposition 1.

We now turn to the proof of Proposition 2. By [9, Theorem 2] we know that there exist constants $\mu_\ell, \eta_\ell \in \mathbb{Q}$ for $\ell \in \mathbb{N}$ such that

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{2r}{k}8^k \sim \frac{(-27)^r}{2^{2/3}\Gamma(2/3)r^{1/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right) \\ + \frac{(-27)^r}{2^{4/3}\Gamma(1/3)r^{2/3}}\left(1 + \sum_{\ell=1}^{\infty}\frac{\eta_\ell}{r^\ell}\right) \qquad (r \to \infty).$$

We are therefore left with showing that the only primes that can divide the denominators of the μ_ℓ and the η_ℓ are 2 and 3. We will use Corollary 2 to accomplish this.

Suppose that $\gamma = -9\alpha\beta$ and define G to be the generating function of $\{u_{r,2r}/\alpha^r\beta^{2r}\}_r$. Define F to be the generating function of the sequence $\{f_r\}_r$ given by

$$f_r := \frac{2^{2/3}\Gamma(2/3)u_{r,2r}}{(-27)^r\alpha^r\beta^{2r}}$$

so that, as $r \to \infty$,

$$f_r \sim \frac{1}{r^{1/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\mu_\ell}{r^\ell}\right)+\frac{\Gamma(2/3)}{2^{2/3}\Gamma(1/3)r^{2/3}}\left(1+\sum_{\ell=1}^{\infty}\frac{\eta_\ell}{r^\ell}\right).$$

A calculation using Maple shows that the generating function F satisfies the linear ordinary differential equation

$$(18x^3-36x^2+18x)F''(x)+(45x^2-54x+9)F'(x)+(9x-5)F(x)=0.$$

We conclude that the function $B(x) := \Psi(F(x))$ satisfies the equation obtained by replacing x with $x+1$. That is,

$$18x^2(x+1)B''(x)+9x(5x+4)B'(x)+(9x+4)B(x)=0.$$

Solving this equation with Maple yields

$$B(x)=C_1x^{-2/3}{}_2F_1\left(-\frac{1}{6},\frac{1}{3};\frac{2}{3};-x\right)+C_2x^{-1/3}{}_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{4}{3};-x\right)$$

for some constants C_1, C_2. However, from (7) and (8) we obtain

$$\Psi(F(x))=\frac{x^{-2/3}}{\Gamma(1/3)}\left(1+\sum_{n=1}^{\infty}c_nx^n\right)+\frac{x^{-1/3}}{2^{2/3}\Gamma(1/3)}\left(1+\sum_{n=1}^{\infty}d_nx^n\right)$$

for certain c_n and d_n. It follows that

$$c_n=[x^n]{}_2F_1\left(-\frac{1}{6},\frac{1}{3};\frac{2}{3};-x\right)=\frac{(-1)^n(-1/6)_n(1/3)_n}{n!(2/3)_n},$$
$$d_n=[x^n]{}_2F_1\left(\frac{1}{6},\frac{2}{3};\frac{4}{3};-x\right)=\frac{(-1)^n(1/6)_n(2/3)_n}{n!(4/3)_n},$$

where $(t)_n = t(t+1)\dots(t+n-1)$ denotes the rising Pochhammer symbol. Since $n!c_n$ and $n!d_n$ have nonnegative valuations at each prime except 2 and 3, we can apply Corollary 2 to obtain the divisibility properties stated for the μ_ℓ and η_ℓ. This completes the proof of Proposition 2.

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