MAT 1341, INTRODUCTION TO LINEAR ALGEBRA, WINTER 2003

Answers to Minitest 2 (Version 1)

Question 1 (2 points). Determine for which value(s) of t the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & t \\ 0 & 1 & 1 \end{pmatrix}$$

is invertible.

- ☐ $t \neq 3$.
- ☒ $t \neq -6$.
- ☐ $t = -1$.
- ☐ $t = -6$.
- ☐ $t = 3$.
- ☐ $t \neq -1$.

Question 2 (2 points). Find the the coordinates of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ relative to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $M_{2,2}$.

- ☐ $[A]_{\mathcal{B}} = [1, -1, 0, 2]$.
- ☒ $[A]_{\mathcal{B}} = [1, 1, 0, 1]$.
- ☐ $[A]_{\mathcal{B}} = [-1, 1, 1, 1]$.
- ☐ $[A]_{\mathcal{B}} = [2, 0, 0, 1]$.
- ☐ $[A]_{\mathcal{B}} = [1, -1, 2, 1]$.
- ☐ $[A]_{\mathcal{B}} = [1, 1, -1, 0]$.

Question 3 (1/2 point each). Let V be an n-dimensional vector space. True or false:

(a) If the vectors $v_1, \ldots, v_m$ span V, then $m < n$. ☐ true ☒ false

(b) Any n vectors which span V are linearly independent. ☒ true ☐ false

(c) Every set of n vectors in V is linearly independent. ☐ true ☒ false

(d) V has a basis consisting of n elements. ☒ true ☐ false

(e) V is spanned by $n-1$ or fewer vectors. ☐ true ☒ false

(f) Any $n+1$ or more vectors in V are linearly dependent. ☒ true ☐ false

Question 4 (1/2 point each). For each of the following sets of vectors, determine whether they are linearly independent or dependent:

(a)	$(0,0,0)$, $(1,0,0)$	☐ independent	☒ dependent
(b)	$(1,0,0)$, $(1,1,0)$, $(1,1,1)$	☒ independent	☐ dependent
(c)	$(4,2,2)$, $(5,1,0)$, $(3,4,2)$, $(-1,0,9)$	☐ independent	☒ dependent
(d)	$(1,5,5)$, $(3,3,2)$	☒ independent	☐ dependent
(e)	$(1,3,4)$, $(1,0,1)$, $(0,1,2)$	☒ independent	☐ dependent
(f)	$(1,3,5)$, $(1,2,3)$, $(1,1,1)$	☐ independent	☒ dependent

Question 5 (3 points). Find a basis and the dimension of the subspace W of $\mathbb{R}^4$ where:

(a) $W = \{(a,b,c,d) \mid a+b+c+d=0\}$.

Answer: We can let b, c, d be arbitrary, then $a=-b-c-d$. So the general solution is:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -b-c-d \\ b \\ c \\ d \end{pmatrix} = b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The dimension is 3, a basis is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) $W = \{(a,b,c,d) \mid a=2b$ and $c=2d\}$.

Answer: We can let b, d be arbitrary, then $a=2b, c=2d$, so the general solution is:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2b \\ b \\ 2d \\ d \end{pmatrix} = b\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

The dimension is 2, a basis is

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Question 6 (3 points). Find a basis and the dimension of span(u_1, u_2, u_3, u_4) in $\mathbf{P}_3(t)$, where

$$\begin{aligned} u_1 &= 2t^3 + 3t^2 + 4t + 5, \\ u_2 &= -1t^3 + 1t^2 + 3t + 0, \\ u_3 &= 1t^3 + 2t^2 + 3t + 3, \\ u_4 &= 2t^3 + 1t^2 + 0t + 4 \end{aligned}$$

Answer: We use row reductions.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ -1 & 1 & 3 & 0 \\ 1 & 2 & 3 & 3 \\ 2 & 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 5 & 10 & 5 \\ 0 & -1 & -2 & -1 \\ 0 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the dimension is 3, and a basis is e.g. $\{2t^3 + 3t^2 + 4t + 5, t^2 + 2t + 1, 1\}$.

Question 7 (4 points). Consider the following matrix:

$$B = \begin{pmatrix} 1 & 5 & 3 & 2 & -1 \\ 4 & -2 & 0 & 1 & 2 \\ 3 & -1 & 1 & 2 & 1 \\ 2 & 6 & 4 & 3 & -1 \\ 4 & 0 & 2 & 3 & 1 \end{pmatrix}$$

(a) What is the rank of B?

Answer:

$$\begin{bmatrix} 1 & 5 & 3 & 2 & -1 \\ 4 & -2 & 0 & 1 & 2 \\ 3 & -1 & 1 & 2 & 1 \\ 2 & 6 & 4 & 3 & -1 \\ 4 & 0 & 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & 2 & -1 \\ 0 & -22 & -12 & -7 & 6 \\ 0 & -16 & -8 & -4 & 4 \\ 0 & -4 & -2 & -1 & 1 \\ 0 & -20 & -10 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 & 2 & -1 \\ 0 & -4 & -2 & -1 & 1 \\ 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3.

(b) Find a subset of the rows of B which forms a basis of the row space of B.

Answer: Cast-out algorithm, using the matrix B^T:

$$\begin{bmatrix} 1 & 4 & 3 & 2 & 4 \\ 5 & -2 & -1 & 6 & 0 \\ 3 & 0 & 1 & 4 & 2 \\ 2 & 1 & 2 & 3 & 3 \\ -1 & 2 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 2 & 4 \\ 0 & -22 & -16 & -4 & -20 \\ 0 & -12 & -8 & -2 & -10 \\ 0 & -7 & -4 & -1 & -5 \\ 0 & 6 & 4 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find that the first three columns are linearly independent. Thus the first three rows of the matrix B are linearly independent; they form a basis of the row space.

(c) Complete the set $u_1 = (1, 3, 1, 5, 3)$, $u_2 = (1, 2, 1, 3, 2)$, $u_3 = (1, 0, 1, 0, 0)$ to a basis of $\mathbb{R}^5$.

Answer: We use row operations to compute a basis of $\mathrm{span}\{u_1, u_2, u_3\}$:

$$\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We can complete this to a basis of $\mathbb{R}^5$ by adding two linearly independent vectors, e.g. $u_4 = (0,0,1,0,0)$ and $u_5 = (0,0,0,0,1)$.

Question 8 (4 points). Recall that a function $F : V \to U$ is linear if (1) for all $v, w \in V$, $F(v+w) = F(v) + F(w)$, and (2) for all $v \in V$, $k \in K$, $F(kv) = kF(v)$.

(a) Show that the following function is linear: $F : \mathbb{R}^2 \to \mathbb{R}^2$, where $F(x,y) = (x+y, x)$.

Answer:

(1) Let $v = (x,y)$ and $w = (x', y')$ be two vectors. Then

$$\begin{aligned} F(v+w) &= F((x,y) + (x',y')) = F(x+x', y+y') = (x+x'+y+y', x+x') \\ F(v) + F(w) &= F(x,y) + F(x',y') = (x+y, x) + (x'+y', x') = (x+y+x'+y', x+x'). \end{aligned}$$

Since the left-hand-side and right-hand-side are equal, we have $F(v+w) = F(v) + F(w)$.

(2) Let $v = (x,y)$ and k a scalar. Then

$$\begin{aligned} F(kv) &= F(kx, ky) = (kx+ky, kx) \\ kF(v) = k(x+y, x) = (k(x+y), yx). \end{aligned}$$

Since the left-hand-side and right-hand-side are equal, we have $F(kv) = kF(v)$.

(b) Show that the following function is not linear: $F : \mathbb{R}^2 \to \mathbb{R}^2$, where $F(x,y) = (xy, x)$. Do this by giving a *concrete* example where one of the above laws ((1) or (2)) is violated.

Answer: We give a counterexample to property (1). Namely, let $v = (0,1)$ and $w = (1,0)$. Then

$$\begin{aligned} F(v+w) &= F(1,1) = (1,1), \\ F(v) + F(w) &= F(0,1) + F(1,0) = (0,0) + (0,1) = (0,1). \end{aligned}$$

Since the two sides are not equal, the equation $F(v+w) = F(v) + F(w)$ does not hold.