MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 8
14.3 #6,10; 14.4 #2,4,10

Problem 14.3 #6 Consider the series

$$\sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{z}{\pi}\right)^n = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!\,k!} \frac{z^n}{\pi^n}$$

(a) We find the radius of convergence by the Cauchy-Hadamard formula:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{n!}{(n-k)!\,k!}\frac{1}{\pi^n}}{\frac{(n+1)!}{(n+1-k)!\,k!}\frac{1}{\pi^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!}{(n-k)!\,k!} \frac{\pi^{n+1}}{\pi^n} \frac{(n+1-k)!\,k!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{n+1-k}{n+1}\pi \right| = \pi$$

(b) We can also find the radius of convergence by reducing the problem to a simpler series. Note that

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Also note that the kth derivative of z_n is

$$\frac{d^k}{dz^k} z^n = n(n-1)\cdots(n-k+1) z^{n-k}.$$

We can therefore start from the geometric series:

$$\sum_{n=k}^{\infty} z^n,$$

whose radius of convergence is 1. We take the kth derivative,

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) z^{n-k},$$

whose radius of convergence is also 1 (by Theorem 3). We multiply by z^k and divide by $k!$ (this does not change the radius of convergence):

$$\sum_{n=k}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} z^n,$$

The radius of convergence is still $|z| < 1$. Finally, we do a substitution: replace z by z/π. The series is

$$\sum_{n=k}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{z}{\pi}\right)^n,$$

which is the same as the desired series. The radius of convergence, after the last substitution, is $|z/\pi| < 1$ or $|z| < \pi$.

Problem 14.3 #10 Consider the series

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!}.$$

(a) We find the radius of convergence by the Cauchy-Hadamard formula. Note that, since all even coefficients are zero, we cannot calculate a_n/a_{n+1} directly. As shown in class, we should first do a substitution $w = z^2$. It also helps to divide the entire series by z (which does not change the radius of convergence). Therefore we have

$$S/z = \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{(2n+1)n!}.$$

We now use the Cauchy-Hadamard formula to find the radius of convergence in terms of w. Namely,

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{(2n+1)n!}}{\frac{(-1)^{n+1}}{(2n+3)(n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n(2n+3)(n+1)n!}{(-1)^{n+1}(2n+1)n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n+3)(n+1)}{(2n+1)} \right| = \infty.$$

The series therefore converges for all w, and hence the original series converges for all z.

(b) We can also find the radius of convergence by reducing the problem to a simpler series. Taking the first derivative of S, we find that

$$\frac{dS}{dz} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n!}.$$

We can substitute $w = -z^2$ so that

$$\sum_{n=0}^{\infty} \frac{w^n}{n!}.$$

This is the familiar series for the exponential function. Its radius of convergence is known to be ∞, hence it converges for all w, and therefore for all z. So the radius of convergence of the original series is ∞.

Problem 14.4 #2 Starting from the familiar geometric series

$$\frac{1}{1-w} = 1 + w + w^2 + \ldots,$$

we can perform a simple substitution $w = z^4$ to obtain the Maclaurin series for:

$$\frac{1}{1-z^4} = 1 + z^4 + z^8 + \ldots.$$

Since the open disc of convergence of the original series is $|w| < 1$, we find after substitution that $|z^4| < 1$, which simplifies to $|z| < 1$. Therefore, the radius of convergence (in terms of z) is 1.

Problem 14.4 #4 We start from the familiar series for e^w:

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}.$$

Substituting $w = -z^2/2$, we get:

$$e^{-z^2/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-z^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} z^{2n} = 1 - \frac{1}{2} z^2 + \frac{1}{4 \cdot 2} z^4 - \frac{1}{8 \cdot 3!} z^6 + \ldots$$

The radius of convergence, in terms of w, is ∞, thus the series converges when $|w| < \infty$, or $|-z^2/2| < \infty$. In other words, the series converges for all z, hence the radius of convergence (in terms of z) is also ∞.

Problem 14.4 #10 To find the Maclaurin series of

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt,$$

we start from the series for $\sin z$:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

The radius of convergence is ∞. We divide by z:

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

Finally, we integrate. The antiderivative of $(\sin z)/z$ is:

$$\mathrm{Si}(z) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} z^{2n+1} = C + z - \frac{z^3}{3 \cdot 3!} + \frac{z^5}{5 \cdot 5!} - \ldots$$

Finally, we determine the constant C: because $\mathrm{Si}(0) = 0$, we have $C = 0$. Therefore

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} z^{2n+1} = z - \frac{z^3}{3 \cdot 3!} + \frac{z^5}{5 \cdot 5!} - \ldots$$