MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 9
15.1 #4,6,12,18

Problem 15.1 #4 We consider the function $f(z) = \frac{1}{z(z-1)}$. It has singularities at $z = 0$ and $z = 1$. We are looking for the Laurent series centered at $z_0 = 0$ whose inner radius of convergence is 0. Due to the singularity at $z = 1$, this series will converge for $0 < |z| < 1$. We have:

$$\frac{1}{z(z-1)} = \frac{-1}{z}\cdot\frac{1}{1-z} = \frac{-1}{z}\underbrace{(1+z+z^2+z^3+\dots)}_{\text{converges for } 0 \leqslant |z| < 1} = -\frac{1}{z}-1-z-z^2-z^3-\dots$$

This is the desired series.

Remark: there is a different Laurent series for $f(z) = \frac{1}{z(z-1)}$ which converges at $1 < |z| < \infty$. It is the following:

$$\frac{1}{z(z-1)} = \frac{-1}{z}\cdot\frac{1}{1-z} = \frac{-1}{z}\underbrace{\left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots\right)}_{\text{converges for } 1 < |z| < \infty} = \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots$$

However, this is not the correct answer to this problem, as the series which converges near 0 was sought.

Problem 15.1 #6 We consider the function $f(z) = \frac{e^{-5z}}{z^5}$. It has only one singularity at $z = 0$. We are looking for the Laurent series centered at $z_0 = 0$ whose inner radius of convergence is 0. Due to the absence of other singularities, this series will converge for $0 < |z| < \infty$. We have:

$$\frac{e^{-5z}}{z^5} = \frac{1}{z^5}e^{-5z} = \frac{1}{z^5}\underbrace{\left(1+(-5z)+\frac{1}{2!}(-5z)^2+\frac{1}{3!}(-5z)^3+\dots\right)}_{\text{converges for all } z}$$
$$= \frac{1}{z^5}-\frac{5}{z^4}+\frac{5^2}{2!z^3}-\frac{5^3}{3!z^2}+\frac{5^4}{4!z}-\frac{5^5}{5!}+\frac{5^6}{6!}z-\dots$$

This is the desired series.

Problem 15.1 #12 We consider the function $f(z) = \frac{z^2-4}{(z-1)^2}$. It has only one singularity at $z = 1$. We are looking for the Laurent series centered at $z_0 = 1$ whose inner radius of convergence is 0. Due to the absence of other singularities, this series will converge for $0 < |z-1| < \infty$. The series is easiest found if we substitute $w = z - 1$. We have:

$$\frac{z^2-4}{(z-1)^2} = \frac{(w+1)^2-4}{w^2} = \frac{w^2+2w-3}{w^2} = \underbrace{1+\frac{2}{w}-\frac{3}{w^2}}_{\text{converges for } |w|>0}$$
$$= 1+\frac{2}{z-1}-\frac{3}{(z-1)^2}$$

This is the desired series. Note that this series consists of finitely many terms; therefore there is no issue of convergence. The series is well-defined for all $z \neq 1$.

Problem 15.1 #18 Consider the function

$$f(z) = \frac{2z-3i}{z^2-3iz-2}.$$

We begin by determining the singularities. We must solve $z^2 - 3iz - 2 = 0$. By the quadratic formula, there are two solutions:

$$z_{1,2} = \frac{3i \pm \sqrt{-9-4(-2)}}{2} = \frac{3i \pm \sqrt{-1}}{2} = \frac{3i \pm i}{2},$$

so the two singularities are $z_1 = 2i$ and $z_2 = i$.

Therefore, the denominator factors as $z^2 - 3iz - 2 = (z - i)(z - 2i)$. Doing partial fractions, we find that

$$\frac{2z-3i}{z^2-3iz-2} = \frac{a}{z-i}+\frac{b}{z-2i}$$

if $a(z - 2i) + b(z - i) = 2z - 3i$. This yields $a = b = 1$, hence

$$\frac{2z-3i}{z^2-3iz-2} = \frac{1}{z-i}+\frac{1}{z-2i}$$

We now seek to develop these fractions into a Laurent series centered at $z_0 = 0$ and converging for $1 < |z| < 2$. We start with the first fraction:

$$\frac{1}{z-i} = \frac{i}{1-\frac{z}{i}}$$

We can change this to a familiar series by substituting $w = \frac{z}{i}$:

$$\frac{i}{1 - \frac{z}{i}} = \frac{i}{1 - w}$$

We would like a series that converges when $1 < |z| < 2$. Since $|w| = |\frac{z}{i}| = |z|$, we therefore want a series for $1 < |w| < 2$. We therefore use:

$$\frac{i}{1 - w} = i(-\frac{1}{w} - \frac{1}{w^2} - \frac{1}{w^3} - \dots)$$

Substituting back for $w = \frac{z}{i}$, we obtain

$$\frac{1}{z - 1} = i(-\frac{i}{z} - \frac{i^2}{z^2} - \frac{i^3}{z^3} - \frac{i^4}{z^4} \dots) = \frac{1}{z} + \frac{i}{z^2} - \frac{1}{z^3} - \frac{i}{z^4} + \dots$$

For the second fraction, we have:

$$\frac{1}{z - 2i} = \frac{i/2}{1 - \frac{z}{2i}}$$

We can change this to a familiar series by substituting $w = \frac{z}{2i}$:

$$\frac{i/2}{1 - \frac{z}{2i}} = \frac{i/2}{1 - w}$$

We would like a series that converges when $1 < |z| < 2$. Since $|w| = |\frac{z}{2i}| = |z/2|$, we therefore want a series that works for $0.5 < |w| < 1$. We therefore use:

$$\frac{i/2}{1 - w} = \frac{i}{2}(1 + w + w^2 + w^3 + \dots)$$

Substituting back for $w = \frac{z}{2i}$, we obtain

$$\frac{1}{z - 1} = \frac{i}{2}(1 + \frac{z}{2i} + \frac{z^2}{(2i)^2} + \frac{z^3}{(2i)^3} + \frac{z^4}{(2i)^4} + \dots) = \frac{i}{2} + \frac{1}{4}z - \frac{i}{8}z^2 - \frac{1}{16}z^3 + \frac{i}{32}z^4 - \dots$$

The Laurent series for the original function is obtained by adding the two parts.

$$f(z) = \frac{2z - 3i}{z^2 - 3iz - 2} = \frac{1}{z} + \frac{i}{z^2} - \frac{1}{z^3} - \frac{i}{z^4} + \dots + \frac{i}{2} + \frac{1}{4}z - \frac{i}{8}z^2 - \frac{1}{16}z^3 + \frac{i}{32}z^4 - \dots$$

when $1 < |z| < 2$.