MAT 3361, INTRODUCTION TO MATHEMATICAL LOGIC, Fall 2004

Handout 1: Rules of Fitch-style natural deduction

Conjunction introduction ($\wedge$I)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A & \\ \vdots & \vdots & \\ n & B & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\mathrm{I}, m, n \end{array} \qquad \begin{array}{l|ll} \vdots & \vdots & \\ m & B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & A \wedge B & \wedge\mathrm{I}, m, n \end{array}$$

Conjunction elimination ($\wedge$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A \wedge B & \\ \vdots & \vdots & \\ n & A & \wedge\mathrm{E}, m \end{array} \qquad \begin{array}{l|ll} \vdots & \vdots & \\ m & A \wedge B & \\ \vdots & \vdots & \\ n & B & \wedge\mathrm{E}, m \end{array}$$

Disjunction Introduction ($\vee$I)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A & \\ \vdots & \vdots & \\ n & A \vee B & \vee\mathrm{I}, m \end{array} \qquad \begin{array}{l|ll} \vdots & \vdots & \\ m & B & \\ \vdots & \vdots & \\ n & A \vee B & \vee\mathrm{I}, m \end{array}$$

Disjunction Elimination ($\vee$E)

$$\begin{array}{l|l|ll} \vdots & \vdots & & \\ m & A \vee B & & \\ m+1 & & \underline{A} & \\ \vdots & & \vdots & \\ n & & \varphi & \\ n+1 & & \underline{B} & \\ \vdots & & \vdots & \\ p & & \varphi & \\ p+1 & \varphi & & \vee\mathrm{E}, m, (m+1)\text{–}n, (n+1)\text{–}p \end{array}$$

Implication Introduction ($\Rightarrow$I)

$$\begin{array}{l|l|ll} \vdots & \vdots & & \\ m & & \underline{A} & \\ \vdots & & \vdots & \\ n & & B & \\ n+1 & A \Rightarrow B & & \Rightarrow\mathrm{I}, m\text{–}n \end{array}$$

Implication Elimination ($\Rightarrow$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A & \\ \vdots & \vdots & \\ n & A \Rightarrow B & \\ \vdots & \vdots & \\ p & B & \Rightarrow\mathrm{E}, m, n \end{array} \qquad \begin{array}{l|ll} \vdots & \vdots & \\ m & A \Rightarrow B & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & B & \Rightarrow\mathrm{E}, m, n \end{array}$$

Negation Introduction ($\neg$I)

$$\begin{array}{l|l|ll} \vdots & \vdots & & \\ m & & \underline{A} & \\ \vdots & & \vdots & \\ n & & \bot & \\ n+1 & \neg A & & \neg\mathrm{I}, m\text{–}n \end{array}$$

Negation Elimination ($\neg$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A & \\ \vdots & \vdots & \\ n & \neg A & \\ \vdots & \vdots & \\ p & \bot & \neg\mathrm{E}, m, n \end{array} \qquad \begin{array}{l|ll} \vdots & \vdots & \\ m & \neg A & \\ \vdots & \vdots & \\ n & A & \\ \vdots & \vdots & \\ p & \bot & \neg\mathrm{E}, m, n \end{array}$$

Contradiction Elimination ($\bot$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & \bot & \\ \vdots & \vdots & \\ n & C & \bot\mathrm{E}, m \end{array}$$

Double negation elimination ($\neg\neg$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & \neg\neg A & \\ \vdots & \vdots & \\ n & A & \neg\neg\mathrm{E}, m \end{array}$$

Repetition (R)

$$\begin{array}{l|l|ll} m & A & & \\ \vdots & \vdots & & \\ \vdots & \cdots & \vdots & \\ n & & A & \mathrm{R}, m \end{array}$$

Forall-introduction ($\forall$I)

$$\begin{array}{l|l|ll} \vdots & \vdots & & \\ m & u & \vdots & \\ \vdots & & \vdots & \\ n & & A(u) & \\ n+1 & \forall x\, A(x) & & \forall\mathrm{I}, m\text{–}n \end{array}$$

Forall-elimination ($\forall$E)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & \forall x\, A(x) & \\ \vdots & \vdots & \\ n & A(t) & \forall\mathrm{E}, m \end{array}$$

Exists-Introduction ($\exists$I)

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & A(t) & \\ \vdots & \vdots & \\ n & \exists x\, A(x) & \exists\mathrm{I}, m \end{array}$$

Exists-Elimination ($\exists$E)

$$\begin{array}{l|l|ll} \vdots & \vdots & & \\ p & \exists x\, A(x) & & \\ m & u & \underline{A(u)} & \\ \vdots & & \vdots & \\ n & & \varphi & \\ n+1 & \varphi & & \exists\mathrm{E}, p, m\text{–}n \end{array}$$

The biconditional

To simplify our formal proof system, we do not introduce any special rules for the connective $\Leftrightarrow$. Instead, we simply regard the formula $A \Leftrightarrow B$ as an *abbreviation* for $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

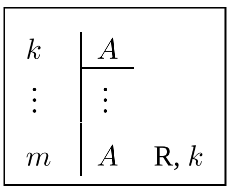
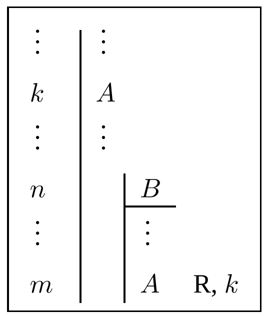
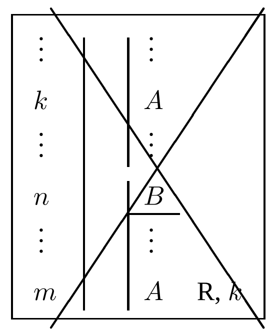
Repetition (R)

Let A be a formula written at line k (either as a hypothesis, or as a formula already proven). Then one can repeat A at line m if:

(1) $k < m$, and

(2) every vertical from line k continues without interruption to line m.

Examples:

This is ok:  This too:  But not this: 

Derived rules

It is possible to create additional rules to be used in natural deduction proofs. These rules are derived from the official rules which are stated above; they can be regarded as "shortcuts". If you want to use such a derived rule, you first have to prove it (i.e., give a separate formal proof which justifies the rule).

One example of a derived rule is De Morgan's law for disjunction,

$$\neg(A \vee B) \vdash \neg A \wedge \neg B.$$

We can give a formal proof of De Morgan's law using only the rules of natural deduction given above:

1	$\neg(A \vee B)$	
2	$\quad A$	
3	$\quad A \vee B$	$\vee$I, 2
4	$\quad \neg(A \vee B)$	R, 1
5	$\quad \bot$	$\neg$E, 3, 4
6	$\neg A$	$\neg$I, 2–5
7	$\quad B$	
8	$\quad A \vee B$	$\vee$I, 7
9	$\quad \neg(A \vee B)$	R, 1
10	$\quad \bot$	$\neg$E, 8, 9
11	$\neg B$	$\neg$I, 7–10
12	$\neg A \wedge \neg B$	$\wedge$I, 6, 11

Having given this formal proof, we can now use De Morgan's law as a *derived rule*, as follows:

$$\begin{array}{l|ll} \vdots & \vdots & \\ m & \neg(A \vee B) & \\ \vdots & \vdots & \\ n & \neg A \wedge \neg B & \text{De Morgan, } m \end{array}$$

Note that there are three other De Morgan's laws, namely

$$\begin{array}{rcl} \neg A \wedge \neg B & \vdash & \neg(A \vee B) \\ \neg(A \wedge B) & \vdash & \neg A \vee \neg B \\ \neg A \vee \neg B & \vdash & \neg(A \wedge B) \end{array}$$

Each of them must be proven separately in natural deduction; thereafter, it can be used as a derived rule.

Problem 1. Give formal proofs of the remaining three laws of De Morgan.

Problem 2. For any proposition φ, let $r(\varphi)$ be the rank of φ and let $c(\varphi)$ be the number of connectives in φ (connectives are $\{\bot, \wedge, \vee, \rightarrow, \leftrightarrow, \neg\}$). (a) Write down recursive definitions of r and c (for r, a definition was already given in class). (b) Prove, by induction, that $r(\varphi) \leqslant c(\varphi)$ for all $\varphi \in \mathrm{PROP}$.

Problem 3. Prove that there exists no $\varphi \in \mathrm{PROP}$ such that the length of φ is 6 symbols.

Problem 4. For the purpose of this problem, we ignore the connectives "$\bot$", "$\rightarrow$" and "$\leftrightarrow$", i.e., we consider propositions built from "$\wedge$", "$\vee$", and "$\neg$" only. The *De Morgan dual* of a proposition φ is defined as follows:

$$\begin{array}{l} dm(p_i) = p_i \\ dm(\mathbf{(}\varphi \wedge \psi\mathbf{)}) = \mathbf{(}dm(\varphi) \vee dm(\psi)\mathbf{)} \\ dm(\mathbf{(}\varphi \vee \psi\mathbf{)}) = \mathbf{(}dm(\varphi) \wedge dm(\psi)\mathbf{)} \\ dm(\mathbf{(}\neg\varphi\mathbf{)}) = \mathbf{(}\neg dm(\varphi)\mathbf{)} \end{array}$$

(a) Let r be the rank function. Prove $r(\varphi) = r(dm(\varphi))$ for all φ.

(b) Let $[\![-]\!]$ be a valuation, and define $[\![-]\!]'$ by $[\![\varphi]\!]' = 1 - [\![dm(\varphi)]\!]$, for all φ. Prove that $[\![-]\!]'$ is a valuation.

(c) A proposition φ is called *satisfiable* if there exists a valuation $[\![-]\!]$ such that $[\![\varphi]\!] = 1$. Prove that φ is satisfiable if and only if $dm(\varphi)$ is not valid.

9. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

10. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

11. $A \wedge (A \vee B) \equiv A$.

12. $A \vee (A \wedge B) \equiv A$.

13. $\vdash \neg(A \wedge (\neg A))$.

14. $A \to (\neg A) \vdash \neg A$.

15. $(A \to B) \wedge (A \to C) \equiv A \to (B \wedge C)$.

16. $A \to B,\ B \to C \vdash A \to C$.

17. $\vdash A \to (B \to A)$.

18. $(A \to B) \to C \vdash B \to C$.

19. $A \to B \vdash (\neg B) \to (\neg A)$.

20. $A \to B \vdash \neg(A \wedge (\neg B))$.

21. $A \wedge (\neg B) \vdash \neg(A \wedge B)$.

22. $(\neg A) \to (\neg B) \vdash B \to (\neg(A \to (\neg B)))$.

23. $\vdash \neg(A \leftrightarrow (\neg A))$.

24. $A \vee B \vdash (B \to A) \to A$.

25. $A \vee B \vdash (\neg B) \to (C \to A)$.

26. $(B \to A) \wedge (A \vee B) \vdash A$.

27. $A \vee B \vdash (\neg A) \to B$.

28. $(\neg A) \vee B \vdash A \to B$.

29. $(\neg A) \vee (\neg B) \vdash \neg(A \wedge B)$.

30. $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$.

31. $(A \vee B) \to C \equiv (A \to C) \wedge (B \to C)$.

32. $(A \to B) \vee (A \to C) \vdash A \to (B \vee C)$.

33. $(A \to C) \vee (B \to C) \vdash (A \wedge B) \to C$.

34. $((\neg A) \vee C) \wedge (B \to C) \vdash (A \to B) \to C$.

35. $\vdash (A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.

36. $(A \to C) \vee (B \to D) \vdash (A \wedge B) \to (C \vee D)$.

37. $A \to B,\ C \to D,\ (\neg B) \vee (\neg D) \vdash (\neg A) \vee (\neg C)$.

38. $A \to (C \vee D),\ (A \vee D) \vee E,\ A \to (\neg C) \vdash D \vee E$.

39. $A \vdash \neg\neg A$.

40. $\neg\neg\neg A \equiv \neg A$.

(NOTE: The $\neg\neg$ rule is required for nos. 41-53.)

41.* $\vdash A \vee (\neg A)$.

42. $(A \to B) \vee C,\ A \to (\neg C) \vdash (B \to C) \to (\neg A)$.

43. $\neg(A \wedge (\neg B)) \vdash A \to B$.

44. $(\neg B) \to (\neg A) \vdash A \to B$.

45.* $A \to B \vdash (\neg A) \vee B$.

46.* $(B \to A) \to A \vdash A \vee B$.

47.* $(A \to B) \to C \vdash A \vee C$.

48.* $(\neg A) \to B \vdash A \vee ((\neg A) \wedge B)$.

49.* $\vdash (A \wedge B) \vee (\neg A) \vee (\neg B)$.

50.* $\neg(A \wedge B) \vdash (\neg A) \vee (\neg B)$.

51.* $\vdash (A \to B) \vee (B \to A)$

52.* $A \to (B \vee C) \vdash (A \to B) \vee (A \to C)$.

53.* $(A \wedge B) \to (C \vee D) \vdash (A \to C) \vee (B \to D)$. (cf.no.36)

54. Prove that if $\varphi \wedge \psi \vdash \theta$ and $\varphi \wedge \theta \vdash \zeta$ then $\varphi \wedge \psi \vdash \zeta$.

55. Prove that if $\varphi \vdash \psi$ then

(i) $\varphi \wedge \theta \vdash \psi \wedge \theta$, (ii) $\varphi \vee \theta \vdash \psi \vee \theta$,

(iii) $\theta \to \varphi \vdash \theta \to \psi$, (iv) $\psi \to \theta \vdash \varphi \to \theta$.

56. Prove that $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$ (Γ is any set of formulae).