MATH 4135/5135: INTRO. TO CATEGORY THEORY, FALL 2005

Midterm Test, November 3, 2005
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This is an "open book" test. You have 90 minutes.

<u>Graduate Students</u>: choose 4 problems.

<u>Undergraduate Students</u>: choose 3 problems.

Problem 1. If $f, g, h : A \to B$ are morphisms in some category $\mathbf{C}$, we define a *3qualizer* of f, g, h to be a limit of the diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, "g" description] \arrow[r, bend right, "h"'] & B. \end{tikzcd}$$

(a) State (without proof) a direct definition of a 3qualizer as a pair (D, d) of an object D and an arrow $d : D \to A$ with a universal property.

Answer: A pair (D, d) is a 3qualizer of f, g, h if $d : D \to A$ and $f \circ d = g \circ d = h \circ d$, and for any other pair (E, e) where $e : E \to A$ and $f \circ e = g \circ e = h \circ e$, there exists a unique morphism $h : E \to D$ such that $e = d \circ h$.

$$\begin{tikzcd} E \arrow[d, dotted, "h"'] \arrow[dr, "e"] & & \\ D \arrow[r, "d"] & A \arrow[r, bend left, "f"] \arrow[r, "g" description] \arrow[r, bend right, "h"'] & B. \end{tikzcd} \tag{1}$$

(b) Assume that $\mathbf{C}$ has equalizers (but not necessarily products). Prove that $\mathbf{C}$ has 3qualizers.

Answer: Let $c : C \to A$ be an equalizer of f and $g : A \to B$. Therefore $f \circ c = g \circ c$. Now let $z : D \to C$ be an equalizer of $g \circ c$ and $h \circ c$. Then $f \circ c \circ z = g \circ c \circ z = h \circ c \circ z$. Let $d = c \circ z$, therefore $f \circ d = g \circ d = h \circ d$.

$$\begin{tikzcd} D \arrow[r, "z"] \arrow[rr, bend left, "d"] & C \arrow[r, "c"] \arrow[rr, bend right, "h \circ c"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \end{tikzcd}$$

We claim that (D, d) has the universal property. So let (E, e) be such that $e : E \to A$ and $f \circ e = g \circ e = h \circ e$. For existence of h, note that, by the universal property of (C, c), there exists $k : E \to C$ such that $c \circ k = e$. But then $h \circ c \circ k = h \circ e = g \circ e = g \circ c \circ k$, so by the universal property of (D, z), there exists $h : E \to D$ such that $z \circ h = k$. But then $e = c \circ k = c \circ z \circ h = d \circ h$, so h makes (1) commute.

For uniqueness, note that c and z are both equalizers, hence monic. Therefore, $d = c \circ z$ is also monic. So if $h' : E \to D$ is another morphism with $e = d \circ h'$, then $d \circ h = d \circ h'$, hence $h = h'$ by the monic property of d.

Problem 2. Let $\mathbf{C}$ be a category. For any (fixed) object $A \in \mathbf{C}$, consider the functor $F_A : \mathbf{C} \to \mathbf{Set}$ defined by $F_A(B) = \hom_{\mathbf{C}}(A, B)$.

(a) What is $F_A(f)$, for a morphism $f : B \to C$ in $\mathbf{C}$?

Answer: $F_A(f) : \hom_{\mathbf{C}}(A, B) \to \hom_{\mathbf{C}}(A, C)$ is defined as $F_A(f)(g) = f \circ g$, for any $g : A \to B$.

(b) Prove: a morphism $f : B \to C$ is monic if and only if for all $A \in |\mathbf{C}|$, $F_A(f) : F_A(B) \to F_A(C)$ is one-to-one.

Answer: Suppose f is monic. To show that $F_A(f)$ is one-to-one, consider $g, h \in F_A(B)$ and suppose $F_A(f)(g) = F_A(f)(h)$.

Then $g, h : A \to B$, and $f \circ g = f \circ h$. Since f is monic, $g = h$, showing that $F_A(f)$ is one-to-one.

Conversely, suppose that for all objects $A \in |\mathbf{C}|$, the function $F_A(f)$ is one-to-one. To show that f is monic, take $g, h : A \to B$ and assume $f \circ g = f \circ h$. But then $F_A(f)(g) = F_A(f)(h)$, hence $g = h$, because $F_A(f)$ is one-to-one. This proves that f is monic.

Problem 3. Let $\mathbf{C}$ be any category, and let $\mathbf{1}$ be the one-object, one-morphism category. Let $F : \mathbf{C} \to \mathbf{1}$ be the unique functor. Prove: F has a right adjoint if and only if $\mathbf{C}$ has a terminal object. Hint: let $*$ be the unique object of $\mathbf{1}$ and consider an isomorphism of hom-sets $\hom_{\mathbf{C}}(A, G(*)) \cong \hom_{\mathbf{1}}(F(A), *)$.

Answer: First, assume that F has a right adjoint $G : \mathbf{1} \to \mathbf{C}$. Let $*$ be the unique object of $\mathbf{1}$, and let $T = G(*)$. We claim that T is a terminal object of $\mathbf{C}$. Indeed, by adjointness, there is a natural isomorphism $\mathbf{C}(A, T) \cong \mathbf{C}(A, G(*)) \cong \mathbf{1}(FA, *)$, but $FA = *$ and the set $\mathbf{1}(FA, *)$ has exactly one element, so that $\mathbf{C}(A, T)$ is a one-element set. This means that for any object A, there exists a unique morphism $h : A \to T$, making T terminal.

Conversely, assume that $\mathbf{C}$ has a terminal object T, and define $G : \mathbf{1} \to \mathbf{C}$ by $G(*) = T$. Then for all objects A, $\mathbf{C}(A, G(*))$ and $\mathbf{1}(FA, *)$ are both 1-element sets, hence they are isomorphic (and the isomorphism is trivially natural). This proves that F and G are adjoints.

Problem 4. A pair (f, g) of morphisms $f : A \to B$ and $g : A \to C$ is called *jointly monic* if for every object H and morphisms $h, k : H \to A$, $f \circ h = f \circ k$ and $g \circ h = g \circ k$ implies $h = k$.

$$\begin{tikzcd}
& H \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & \\
& A \arrow[dl, "f"'] \arrow[dr, "g"] & \\
B & & C
\end{tikzcd}$$

Prove: if

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\
C \arrow[r, "n"] & D
\end{tikzcd}$$

is a pullback, then (f, g) is jointly monic.

Answer: Assume a pullback is given as above, and consider $h, k : H \to A$ such that $f \circ h = f \circ k$ and $g \circ h = g \circ k$. Let $f' = f \circ h = f \circ k$, and $g' = g \circ h = g \circ k$. Then each of the following diagram commutes by definition:

$$\begin{tikzcd}
H \arrow[dr, "h"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g'"'] & & \\
& A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\
& C \arrow[r, "n"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[dr, "k"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g'"'] & & \\
& A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\
& C \arrow[r, "n"] & D
\end{tikzcd}$$

On the other hand, by the defining property of a pullback, there exists a unique h making this diagram commute, so $h = k$. This shows that (f, g) is jointly monic.

Problem 5. Let $\mathbb{R}$ be the set of real numbers, regarded as an object in the category **Set**. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by: $f(x) = x^2$, $g(y) = 1 - y^2$. Find D, d_1, and d_2 such that the following is a pullback:

$$\begin{tikzcd} D \arrow[r, "d_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "g"] \\ \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Answer: By the standard formula for pullbacks in set, we can let

$$\begin{aligned} D &= \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid f(x) = g(y)\} \\ &= \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = 1 - y^2\} \\ &= \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}. \end{aligned}$$

Therefore, $D \subseteq \mathbb{R} \times \mathbb{R}$ is the unit circle, and $d_1, d_2 : D \to \mathbb{R}$ are the projections onto the first and second coordinate, respectively: $d_1(x,y) = x$, $d_2(x,y) = y$.