

MATH 4135/5135: INTRO. TO CATEGORY THEORY, FALL
2005

Midterm Test, November 3, 2005
Prof. P. Selinger

This is an “open book” test. You have 90 minutes.

Graduate Students: choose 4 problems.

Undergraduate Students: choose 3 problems.

Problem 1. If $f, g, h : A \rightarrow B$ are morphisms in some category \mathbf{C} , we define a *3qualizer* of f, g, h to be a limit of the diagram

$$\begin{array}{ccc} & f & \\ & \curvearrowright & \\ A & \xrightarrow{g \quad +} & B \\ & \curvearrowleft & \\ & h & \end{array}$$

- (a) State (without proof) a direct definition of a 3qualizer as a pair (D, d) of an object D and an arrow $d : D \rightarrow A$ with a universal property.

Answer: A pair (D, d) is a 3qualizer of f, g, h if $d : D \rightarrow A$ and $f \circ d = g \circ d = h \circ d$, and for any other pair (E, e) where $e : E \rightarrow A$ and $f \circ e = g \circ e = h \circ e$, there exists a unique morphism $h : E \rightarrow D$ such that $e = d \circ h$.

$$\begin{array}{ccc} E & & \\ \downarrow h & \searrow e & \\ D & \xrightarrow{d} & A \end{array} \quad \begin{array}{ccc} & f & \\ & \curvearrowright & \\ & g \quad + & \\ & \curvearrowleft & \\ & h & \end{array} \rightarrow B. \quad (1)$$

- (b) Assume that \mathbf{C} has equalizers (but not necessarily products). Prove that \mathbf{C} has 3qualizers.

Answer: Let $c : C \rightarrow A$ be an equalizer of f and $g : A \rightarrow B$. Therefore $f \circ c = g \circ c$. Now let $z : D \rightarrow C$ be an equalizer of $g \circ c$ and $h \circ c$. Then $f \circ c \circ z = g \circ c \circ z = h \circ c \circ z$. Let $d = c \circ z$, therefore $f \circ d = g \circ d = h \circ d$.

$$\begin{array}{ccccc} & & d & & \\ & & \curvearrowright & & \\ D & \xrightarrow{z} & C & \xrightarrow{c} & A \end{array} \begin{array}{ccc} & f & \\ & \curvearrowright & \\ & + & \\ & \curvearrowleft & \\ & g & \end{array} \rightarrow B. \\ & & & & \text{hoc} \end{array}$$

We claim that (D, d) has the universal property. So let (E, e) be such that $e : E \rightarrow A$ and $f \circ e = g \circ e = h \circ e$. For existence of h , note that, by the universal property of (C, c) , there exists $k : E \rightarrow C$ such that $c \circ k = e$. But then $h \circ c \circ k = h \circ e = g \circ e = g \circ c \circ k$, so by the universal property of (D, z) , there exists $h : E \rightarrow D$ such that $z \circ h = k$. But then $e = c \circ k = c \circ z \circ h = d \circ h$, so h makes (1) commute.

For uniqueness, note that c and z are both equalizers, hence monic. Therefore, $d = c \circ z$ is also monic. So if $h' : E \rightarrow D$ is another morphism with $e = d \circ h'$, then $d \circ h = d \circ h'$, hence $h = h'$ by the monic property of d .

Problem 2. Let \mathbf{C} be a category. For any (fixed) object $A \in \mathbf{C}$, consider the functor $F_A : \mathbf{C} \rightarrow \mathbf{Set}$ defined by $F_A(B) = \text{hom}_{\mathbf{C}}(A, B)$.

- (a) What is $F_A(f)$, for a morphism $f : B \rightarrow C$ in \mathbf{C} ?

Answer: $F_A(f) : \text{hom}_{\mathbf{C}}(A, B) \rightarrow \text{hom}_{\mathbf{C}}(A, C)$ is defined as $F_A(f)(g) = f \circ g$, for any $g : A \rightarrow B$.

- (b) Prove: a morphism $f : B \rightarrow C$ is monic if and only if for all $A \in |\mathbf{C}|$, $F_A(f) : F_A(B) \rightarrow F_A(C)$ is one-to-one.

Answer: Suppose f is monic. To show that $F_A(f)$ is one-to-one, consider $g, h \in F_A(B)$ and suppose $F_A(f)(g) = F_A(f)(h)$.

Then $g, h : A \rightarrow B$, and $f \circ g = f \circ h$. Since f is monic, $g = h$, showing that $F_A(f)$ is one-to-one.

Conversely, suppose that for all objects $A \in |\mathbf{C}|$, the function $F_A(f)$ is one-to-one. To show that f is monic, take $g, h : A \rightarrow B$ and assume $f \circ g = f \circ h$. But then $F_A(f)(g) = F_A(f)(h)$, hence $g = h$, because $F_A(f)$ is one-to-one. This proves that f is monic.

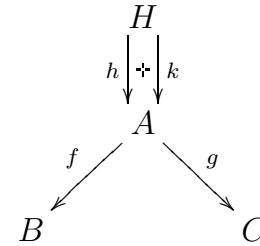
Problem 3. Let \mathbf{C} be any category, and let $\mathbf{1}$ be the one-object, one-morphism category. Let $F : \mathbf{C} \rightarrow \mathbf{1}$ be the unique functor. Prove: F has a right adjoint if and only if \mathbf{C} has a terminal object. Hint: let $*$ be the unique object of $\mathbf{1}$ and consider an isomorphism of hom-sets $\text{hom}_{\mathbf{C}}(A, G(*)) \cong \text{hom}_{\mathbf{1}}(F(A), *)$.

Answer: First, assume that F has a right adjoint $G : \mathbf{1} \rightarrow \mathbf{C}$. Let $*$ be the unique object of $\mathbf{1}$, and let $T = G(*)$. We claim that T is a terminal object of \mathbf{C} . Indeed, by adjointness, there is a natural isomorphism $\mathbf{C}(A, T) \cong \mathbf{C}(A, G(*)) \cong \mathbf{1}(FA, *)$, but $FA = *$ and the set $\mathbf{1}(FA, *)$ has exactly one element, so that $\mathbf{C}(A, T)$ is a one-element set. This means that for any object A , there exists a unique morphism $h : A \rightarrow T$, making T terminal.

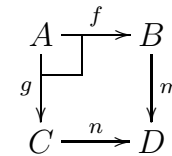
Conversely, assume that \mathbf{C} has a terminal object T , and define $G : \mathbf{1} \rightarrow \mathbf{C}$ by $G(*) = T$. Then for all objects A , $\mathbf{C}(A, G(*))$ and $\mathbf{1}(FA, *)$ are both 1-element sets, hence they are isomorphic (and the isomorphism is trivially natural). This proves that F and G are adjoints.

Problem 4. A pair (f, g) of morphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ is called *jointly monic* if for every object H and morphisms $h, k : H \rightarrow A$, $f \circ h = f \circ k$ and $g \circ h = g \circ k$ implies $h = k$.

$H \rightarrow A$, $f \circ h = f \circ k$ and $g \circ h = g \circ k$ implies $h = k$.

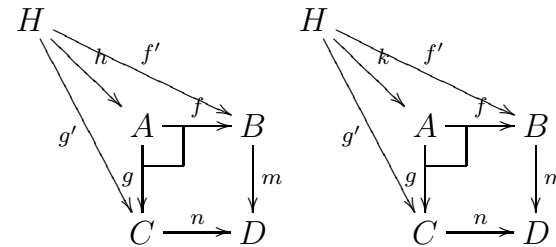


Prove: if



is a pullback, then (f, g) is jointly monic.

Answer: Assume a pullback is given as above, and consider $h, k : H \rightarrow A$ such that $f \circ h = f \circ k$ and $g \circ h = g \circ k$. Let $f' = f \circ h = f \circ k$, and $g' = g \circ h = g \circ k$. Then each of the following diagram commutes by definition:



On the other hand, by the defining property of a pullback, there exists a unique h making this diagram commute, so $h = k$. This shows that (f, g) is jointly monic.

Problem 5. Let \mathbb{R} be the set of real numbers, regarded as an object in the category **Set**. Consider the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ defined

by: $f(x) = x^2$, $g(y) = 1 - y^2$. Find D , d_1 , and d_2 such that the following is a pullback:

$$\begin{array}{ccc}
 D & \xrightarrow{d_2} & \mathbb{R} \\
 d_1 \downarrow & \lrcorner & \downarrow g \\
 \mathbb{R} & \xrightarrow{f} & \mathbb{R}
 \end{array}$$

Answer: By the standard formula for pullbacks in set, we can let

$$\begin{aligned}
 D &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid f(x) = g(y)\} \\
 &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 = 1 - y^2\} \\
 &= \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}.
 \end{aligned}$$

Therefore, $D \subseteq \mathbb{R} \times \mathbb{R}$ is the unit circle, and $d_1, d_2 : D \rightarrow \mathbb{R}$ are the projections onto the first and second coordinate, respectively:
 $d_1(x, y) = x$, $d_2(x, y) = y$.