MATH 285, HONORS MULTIVARIABLE CALCULUS, FALL 1999

Answers to the Second Midterm

Problem 1 (8 points) Use a tangent plane approximation to approximate $(0.99)^3 e^{0.01}$.

Answer: Let $f(x,y) = x^3 e^y$. The tangent plane approximation for f near a point (x_0, y_0) is

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y,$$

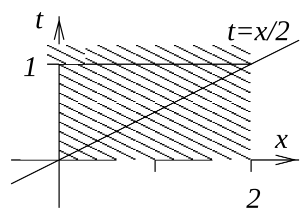
where $\Delta x = x - x_0$ and $\Delta y = y - y_0$. We calculate $f_x(x,y) = 3x^2 e^y$ and $f_y(x,y) = x^3 e^y$. If we take $(x_0, y_0) = (1, 0)$, we get:

$$\begin{aligned} f(0.99, 0.01) &\approx f(1,0) + f_x(1,0)\cdot(-0.01) + f_y(1,0)\cdot 0.01 \\ &= 1 - 3\cdot 0.01 + 1\cdot 0.01 \\ &= 0.98. \end{aligned}$$

Problem 2 (8 points) Find the average value of the function $f(x) = \int_{x/2}^{1} \sin(t^2)\,dt$ on the interval $[0, 2]$.

Answer: The average value of $f(x)$ on the interval $[0,2]$ is

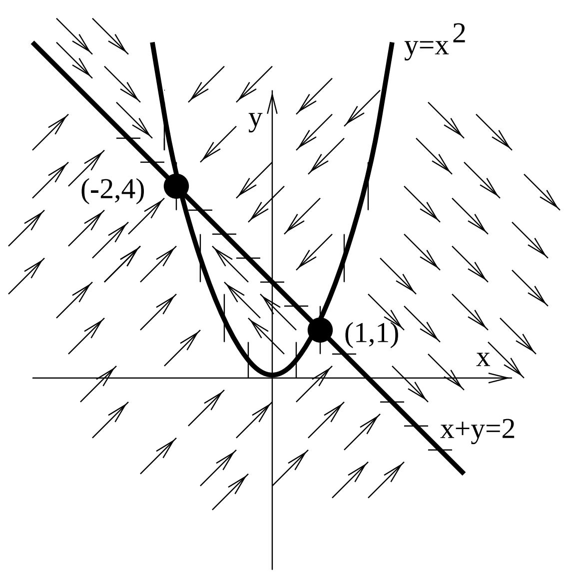
$$\begin{aligned} \tfrac{1}{2}\int_0^2 f(x)\,dx &= \tfrac{1}{2}\int_0^2 \int_{x/2}^1 \sin(t^2)\,dt\,dx \\ &= \tfrac{1}{2}\int_0^1 \int_0^{2t} \sin(t^2)\,dx\,dt \\ &= \tfrac{1}{2}\int_0^1 2t\sin(t^2)\,dt \\ &= \tfrac{1}{2}\left[-\cos(t^2)\right]_0^1 \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos 1. \end{aligned}$$



Problem 3 Suppose $f(x,y)$ is a function such that $\nabla f = \langle x^2 - y, -x - y + 2\rangle$.

(a) **(5 points)** Use the nullcline method to sketch the vector field ∇f.

Answer:



(b) **(5 points)** Determine all local maxima, minima, and saddle points of f. Does f have a global maximum or minimum?

Answer: The critical points of f occur where $f_x = 0$ and $f_y = 0$. In the above sketch, we see that this happens at $(x, y) = (-2, 4)$ and $(x, y) = (1, 1)$. We do the second derivative test: $D = f_{xx}f_{yy} - f_{xy}^2 = 2x \cdot (-1) - (-1)^2 = 1 - 2x$. We find that D is negative at $(x, y) = (1, 1)$, making $(1, 1)$ a saddle point. Also, D is positive at $(x, y) = (-2, 4)$, making $(-2, 4)$ into a local maximum or minimum. Since $f_x x(-2, 4) = -4$ is negative, we find that $f(-2, 4)$ is a local maximum. This can also be seen by looking at the above sketch and using the fact that the gradient always points in the direction of greatest increase.

f has no global maxima or minima. If we keep $y = 0$ fixed, we find that $df/dx = x^2$, thus $f(x, 0) = x^3/3 + C$, thus $\lim_{x\to\infty} f(x, 0) = +\infty$ and $\lim_{x\to-\infty} f(x, 0) = -\infty$.

Problem 4 (10 points) Franny says: “I know a differentiable function $f(x, y)$ such that

$$\frac{\partial}{\partial x} f(x, y) = xe^{x^2y^2} \qquad \text{and} \qquad \frac{\partial}{\partial y} f(x, y) = ye^{x^2y^2}.$$

Zooey says: “And I know a differentiable function $g(x, y)$ such that

$$\frac{\partial}{\partial x} g(x, y) = ye^{x^2y^2} \qquad \text{and} \qquad \frac{\partial}{\partial y} g(x, y) = xe^{x^2y^2}.$$

One of them is lying. Which one, and how do you know?

Answer: One way to check this is by Clairaut’s Theorem. Checking Franny’s function, we have

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x, y) = xe^{x^2y^2} \cdot 2yx^2, \qquad \text{and} \qquad \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x, y) = ye^{x^2y^2} \cdot 2xy^2.$$

If Franny really had such a function $f(x, y)$, then by Clairaut’s Theorem, the two expressions would have to be equal, but they are not. So Franny is lying. On the other hand, for Zooey’s function, we get

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} g(x, y) = e^{x^2y^2} + ye^{x^2y^2} \cdot 2yx^2, \qquad \text{and} \qquad \frac{\partial}{\partial x}\frac{\partial}{\partial y} g(x, y) = e^{x^2y^2} + xe^{x^2y^2} \cdot 2xy^2.$$

These functions are indeed equal, which makes Zooey’s claim plausible. You did not have to find the function g, but in case you’re curious, here is a possible definition:

$$g(x, y) = \int_0^{xy} e^{t^2}\, dt.$$

Problem 5 Consider the function f which is defined as follows:

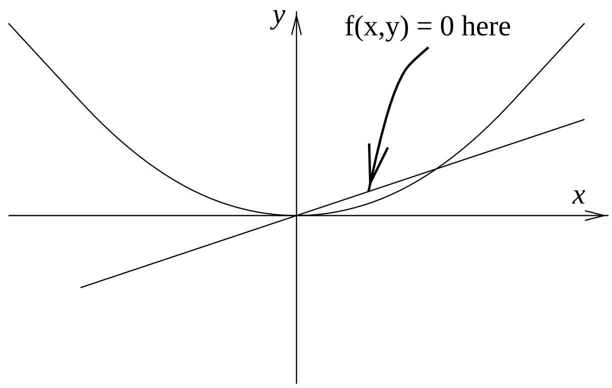
$$f(x, y) = \begin{cases} 1 & \text{if } x \neq 0 \text{ and } y = x^2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) **(4 points)** Is f differentiable at $(0, 0)$?

Answer: f is not continuous at $(0, 0)$, because $\lim_{x\to 0} f(x, x^2) = \lim_{x\to 0} 1 = 1$, whereas $f(0, 0) = 0$. Since f is not continuous, it is certainly not differentiable.

(b) **(10 points)** Prove that for every unit vector $\vec{\mathbf{u}}$, the directional derivative $D_{\vec{\mathbf{u}}}f(0,0)$ exists.

Answer: Informally, the reason is that any fixed line through the origin has a small stretch near (0,0) where f is constantly zero.



More formally: Let $\vec{\mathbf{u}} = \langle a, b\rangle$ be any fixed unit vector. The directional derivative $D_{\vec{\mathbf{u}}}f(0,0)$ is defined as the limit

$$\lim_{h\to 0} \frac{f(ha,hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h}.$$

We consider two cases: Case 1: $a = 0$. In this case, $f(ha,hb) = f(0,hb) = 0$ for all h, thus, the above limit exists and is equal to zero. Case 2: $a \neq 0$. By definition of f, we have $f(ha,hb) = 1$ only when $hb = (ha)^2$ and $ha \neq 0$, i.e., when $hb = h^2a^2$, which happens only at $h = \frac{b}{a^2}$. Thus, when h is sufficiently small (namely, when $|h| < |\frac{b}{a^2}|$), then we have $f(ha,hb) = 0$. It follows that $\lim_{h\to 0} \frac{f(ha,hb)}{h} = 0$.

In either case, the limit exists.