**MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999**

**Answers to Problem Set 10**

**Problem 1** The given $F$ is clearly a function class. Also, for all sets $z$, one has $F(F(z)) = z$, and thus $F$ is one-to-one and onto the class of sets.

Here are some other examples of bijections of the universe onto itself. In the definition of $H$, $A$ is a fixed arbitrary set.

$$\begin{array}{lcll} G(n) & = & \{n\} & \text{if } n \in \omega, n \geqslant 1, \\ G(\{n\}) & = & n & \text{if } n \in \omega, n \geqslant 1, \\ G(x) & = & x & \text{in all other cases,} \end{array} \qquad \begin{array}{lcll} H(x) & = & x \cup \{A\} & \text{if } A \notin x, \\ H(x) & = & x - \{A\} & \text{if } A \in x. \end{array}$$

**Problem 2** For any set $x$, one has $x \in' F^{-1}(\emptyset)$ iff $x \in F(F^{-1}(\emptyset))$ iff $x \in \emptyset$. This holds for no $x$. Moreover, if $F$ is the bijection from Problem 1, then $0 \in 1 = F(0)$, thus $0 \in' 0$.

**Problem 3** Let $A$ and $B$ be any sets such that for all $x$, $x \in' A \iff x \in' B$. This means, $x \in F(A) \iff x \in F(B)$. By extensionality (for $\mathscr{U}$), this implies $F(A) = F(B)$, hence, since $F$ is one-to-one, $A = B$.

**Problem 4** We have

$$\begin{array}{rcll} x \in' B & \iff & x \in F(B) & \text{by definition of } \in' \\ & \iff & x \in c & \text{by definition of } B \\ & \iff & \exists b\,(x \in F(b) \wedge b \in F(A)) & \text{by definition of union} \\ & \iff & \exists b\,(x \in' b \wedge b \in' A) & \text{by definition of } \in'. \end{array}$$

Thus we have shown

$$\forall A\, \exists B\, \forall x\, [x \in' B \iff \exists b\,(x \in' b \wedge b \in' A)],$$

which is precisely the union axiom for $\mathscr{U}'$.

**Problem 5** We have $c = \{F^{-1}(z) \mid z \subseteq F(A)\} = \{F^{-1}(z) \mid z \in \mathscr{P}(F(A))\}$. This is a set by the powerset axiom and replacement. Moreover, we have $x \in' B$ iff $x \in F(B)$ iff $x \in c$ iff $F(x) \subseteq F(A)$ iff $x \subseteq' A$. Thus, we have shown that

$$\forall A\, \exists B\, \forall x\, (x \in' B \iff x \subseteq' A),$$

which is precisely the power set axiom for $\mathscr{U}'$.

**Problem 6** We verify the three properties:

$$\begin{array}{rcll} z \in' x \cup' y & \iff & z \in F(x \cup' y) & \text{by definition of } \in' \\ & \iff & z \in F(x) \cup F(y) & \text{by definition of } \cup' \\ & \iff & z \in F(x) \vee z \in F(y) & \text{by definition of union} \\ & \iff & z \in' x \vee z \in' y & \text{by definition of } \in'. \end{array}$$

The reasoning for $\cap'$ is entirely analogous. Finally

$$\begin{array}{rcl} z \in' \{x\}' & \iff & z \in F(\{x\}') \\ & \iff & z \in \{x\} \\ & \iff & z = x. \end{array}$$

**Problem 7** Since $\eta$ is the range of $f$, we have $\emptyset' = f(0) \in \eta = F(A)$, and thus $\emptyset' \in' A$. Also, for any $a$, if $a \in' A$, then $a \in F(A) = \eta$, thus $a = f(n)$ for some $n$. Then $a \cup' \{a\}' = f(n^+)$, and thus $a \cup' sa' \in \eta = F(A)$, which implies $a \cup' sa' \in' A$.

**Problem 8** Suppose that $x \in A_1$. Let $u$ be such that $x = F(u)$. Then $u \in F(A)$ by definition of $A_1$ (recall that $F$ is a bijection). This implies $u \in' A$, and hence $u \neq \emptyset'$ by assumption. It follows that $x = F(u) \neq F(\emptyset') = \emptyset$. This proves the first claim.

Now assume that $x, y \in A_1$ and $x \neq y$. Let $u$ and $v$ be such that $x = F(u)$ and $y = F(v)$. Then $u, v \in F(A)$ by definition of $A_1$, hence $u, v \in' A$. Since $u \neq v$, we have $u \cap' v = \emptyset'$ by assumption. By definition of $\cap'$ and $\emptyset'$, this means $F^{-1}(F(u) \cap F(v)) = F^{-1}(\emptyset)$, hence $F(u) \cap F(v) = \emptyset$ since $F^{-1}$ is one-to-one. But this means $x \cap y = \emptyset$, as desired.

**Problem 9** We have $C \cap' x = \{w\}'$ iff $F^{-1}(F(C) \cap F(x)) = F^{-1}(\{w\})$ iff $F(C) \cap F(x) = \{w\}$, by definition of $\cap'$, $\{w\}'$, and the fact that $F$ is a bijection. But $F(C) = C_1$, so the first claim follows. Further, it follows directly from the definition of $A_1$, and the fact that $F$ is a bijection, that $F(x) \in A_1$ iff $x \in F(A)$ iff $x \in' A$.

**Problem 10** In Problem 2, we have seen that the universe $\mathscr{U}'$ in question satisfies $0 \in' 0$. But by Theorem 7X, this contradicts regularity. Thus, regularity cannot hold in the universe $\mathscr{U}'$.