MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 5

Problem 4.2 Suppose a is a transitive set. Then $\bigcup(a^+) = a \subseteq a^+$ by Theorem 4E. This shows that a^+ is transitive.

Problem 4.3

(a) Suppose a is a transitive set. To show that $\mathscr{P}a$ is transitive, take any $x \in y \in \mathscr{P}a$; it suffices to show that $x \in \mathscr{P}a$. We know $x \in y \subseteq a$, hence $x \in a$. Since a is transitive, this implies $x \subseteq a$, thus $x \in \mathscr{P}a$, as desired.

(b) Suppose $\mathscr{P}a$ is a transitive set. To show that a is transitive, take any $y \in a$. It suffices to show that $y \subseteq a$. We know that $\{y\} \subseteq a$, hence $\{y\} \in \mathscr{P}a$. Since $\mathscr{P}a$ is transitive, this implies $\{y\} \subseteq \mathscr{P}a$, hence $y \in \mathscr{P}a$, thus $y \subseteq a$, as desired.

Problem 4.4 Suppose that a is a transitive set. To show that $\bigcup a$ is transitive, take any $y \in \bigcup a$. It will suffice to show that $y \subseteq \bigcup a$. We know that $y \in z \in a$ for some z, by definition of union. Thus $y \in a$ by transitivity of a, which implies $y \subseteq \bigcup a$, as desired.

Problem 4.6 Suppose $\bigcup(a^+) = a$. To prove that a is transitive, observe that $a \subseteq a^+$, hence $\bigcup a \subseteq \bigcup(a^+) = a$ by Problem 2.4. This shows a is transitive.

Problem 4.8 Assume $f : A \to A$ is one-to-one and $c \in A - \operatorname{ran} f$. Define $h : \omega \to A$ by recursion:

$$h(0) = c,$$
$$h(n^+) = f(h(n)).$$

Remark: First observe that $h(n^+) \neq h(0)$ for all $n \in \omega$, since $h(n^+)$ is in the range of f, whereas $h(0)$ is not. Also recall from Theorem 4C that every natural number is either 0 or a successor.

To show that h is one-to-one, we have to show that $h(n) = h(m)$ implies $n = m$, for all $n, m \in \omega$. We prove this by induction on n. So let

$$T = \{n \in \omega \mid (\forall m \in \omega)\, h(n) = h(m) \Rightarrow n = m\}.$$

We show that T is inductive. *Base case:* Suppose $h(0) = h(m)$. By the above remark, m is not a successor, hence $m = 0$. *Induction step:* Assume $n \in T$. To show that $n^+ \in T$, assume $h(n^+) = h(m)$. By the remark, $m \neq 0$, so $m = k^+$ for some $k \in \omega$. Then $h(n^+) = h(k^+)$ implies $f(h(n)) = f(h(k))$, which implies $h(n) = h(k)$, since f is one-to-one. By induction hypothesis, $h(n) = h(k)$ implies $n = k$, and thus $n^+ = k^+ = m$. This shows that T is inductive, and thus that h is one-to-one.

Problem 4.9 To show that $C^* = C_*$, we show each inclusion seperately. Let

$$\mathscr{B} = \{X \mid A \subseteq X \subseteq B \wedge f[\![X]\!] \subseteq X\}.$$

To prove the left-to-right inclusion, we first claim that $f[\![C_*]\!] \subseteq C_*$. To prove this, consider any $y \in f[\![C_*]\!]$. Then $y = f(x)$ for some $x \in C_*$. Since $C_* = \bigcup_{i\in\omega} h(i)$, this implies $x \in h(i)$ for some $i \in \omega$, by definition of union. It follows that $y = f(x) \in f[\![h(i)]\!] \subseteq h(i^+) \subseteq C_*$. Since y was arbitrary, this proves our first claim.

Also, note that $A = h(0) \subseteq C_* \subseteq B$. Thus, C_* is a member of the set $\mathscr{B}$. But $C^* = \bigcap \mathscr{B}$, and hence $C^* \subseteq C_*$, which proves the first inclusion.

Before we prove the right-to-left inclusion, notice that for all $X, Y \subseteq B$, $X \subseteq Y$ implies $f[\![X]\!] \subseteq f[\![Y]\!]$. This is proved as in Theorem 3K.

We claim that $h(n) \subseteq C^*$ for all $n \in \omega$. We prove this by induction: *Base case:* Since $A \subseteq X$ holds for all $X \in \mathscr{B}$, we have $A \subseteq \bigcap \mathscr{B}$, and thus $h(0) \subseteq C^*$. *Induction step:* Suppose $h(n) \subseteq C^*$. Then for all $X \in \mathscr{B}$, one has $h(n) \subseteq X$, and hence, by the above remark, $f[\![h(n)]\!] \subseteq f[\![X]\!] \subseteq X$. Since this holds for all $X \in \mathscr{B}$, we also have $f[\![h(n)]\!] \subseteq \bigcap \mathscr{B} = C^*$. We already know $h(n) \subseteq C^*$, and thus $h(n^+) = h(n) \cup f[\![h(n)]\!] \subseteq C^*$. This finishes the induction step.

We have shown that $h(n) \subseteq C^*$ for all $n \in \omega$. This implies the desired inclusion $C_* = \bigcup_{i\in\omega} h(i) \subseteq C^*$. So the proof is finished.

Problem 4.13 Suppose to the contrary that there are natural numbers $m, n \neq 0$ such that $m \cdot n = 0$. Then $m = k^+$ and $n = l^+$ for some $k, l \in \omega$. We calculate

$$\begin{aligned} m \cdot n &= k^+ \cdot l^+ \\ &= k^+ \cdot l + k^+ && \text{by (M2)} \\ &= (k^+ \cdot l + k)^+ && \text{by (A2),} \end{aligned}$$

Thus, $m \cdot n$ is a successor, a contradiction.

Problem 4.17 First notice that for all $m \in \omega$, one has $m \cdot 1 = m \cdot 0^+ = m \cdot 0 + m = 0 + m = m + 0 = m$, by definition of 1, (M2), (M1), commutativity of +, and (A1).

We now prove $m^{n+p} = m^n \cdot m^p$ by induction on p. *Base case:* For $p = 0$, we have

$$\begin{aligned} m^{n+0} &= m^n && \text{by (A1)} \\ &= m^n \cdot 1 && \text{by the above remark} \\ &= m^n \cdot m^0 && \text{by (E1)} \end{aligned}$$

Induction step: Suppose the claim holds for p. To show that it holds for p^+, we calculate

$$\begin{aligned} m^{n+p^+} &= m^{(n+p)^+} && \text{by (A2)} \\ &= m^{n+p} \cdot m && \text{by (E2)} \\ &= (m^n \cdot m^p) \cdot m && \text{by ind. hyp.} \\ &= m^n \cdot (m^p \cdot m) && \text{by associativity of multiplication} \\ &= m^n \cdot m^{p^+} && \text{by (E2).} \end{aligned}$$